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ELECTRICAL NOTES

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IX. OSCILLATIONS WITH A SPLIT-ANODE MAGNETRON

[Received 27 June 1936; in revised form 19 August 1936]

1. Introduction

In applying the methods of my last note† to split-anode magnetrons I have continued to neglect the inequality of emission velocity and to assume that most of the electrons are caught on the filament when they return. This restricts us to emissions of the order of a microampere, and also cuts across another assumption, necessary for a simple theory, that the paths of the electrons are nearly circles. I therefore regard this note as answering a qualitative question, whether known electrical principles can account for the maintenance of *any* short-waved oscillations by a split-anode magnetron, by taking a strict theory and stretching the conditions of its application. The result appears, within its limits, to be encouraging.

2. Motion of electrons between a filament and a split anode

The currents are supposed to be so small that the electric and magnetic field of the electrons themselves has no appreciable effect on their motion, the electrons being taken to move in a field of scalar electromagnetic potential V, on which a uniform magnetic field H is superposed parallel to the filament. Let $-e$ be the charge and m the mass of an electron,

$$\omega = He/mc. \tag{1}$$

The equations of motion are

$$\ddot{x}+\omega\dot{y} = \frac{e}{m}\frac{\partial V}{\partial x}, \qquad \ddot{y}-\omega\dot{x} = \frac{e}{m}\frac{\partial V}{\partial y}.$$

Let the motion in an assigned static field be known, and let $x(t)$, $y(t)$ be the coordinates at time t of an electron leaving a given point at time t_0 with given velocities in the plane of x, y. Let V now denote the scalar electromagnetic potential of a small field superposed on the static field, so that V is a given function of x, y and t, and let $X(t)$, $Y(t)$ be the result of substituting $x(t)$, $y(t)$ for x, y in $\partial V/\partial x$,

† See above, pp. 210–213.

$\partial V/\partial y$ respectively. Then if x, y are now the small changes of coordinate at time t,

$$\ddot{x}+i\ddot{y}-i\omega(\dot{x}+i\dot{y}) = (e/m)[X(t)+iY(t)]$$

and $x = y = \dot{x} = \dot{y} = 0$ when $t = t_0$. Hence

$$x+iy = \frac{ie}{m\omega}\int\limits_{t_0}^{t} [X(\tau)+iY(\tau)][1-\exp i(\omega t-\omega\tau)]\,d\tau. \tag{2}$$

Let a be the radius of the filament, small and sometimes taken as zero, b the radius of the anode, and let the anode be divided by a fine cut at polar angles $\pm\frac{1}{2}\pi$. Let all the electrons be emitted normally with the same velocity. Then in the steady motion, with the same constant potential applied to both halves of the anode, all are turned back at the same radius, and we shall examine the effects produced when that radius is nearly b.

Let the disturbance consist of potentials $\pm\frac{1}{2}V$ applied to the right- and left-hand halves of the anode, where V is in general a small function of t, but such that the wave-length in air of the corresponding electric oscillations is much greater than b. The effect of the filament on the field so added will be neglected, so that the added potential at a point $z = x+iy$ inside the circle $|z| = b$ is the real part of the function

$$W = -\frac{iV}{\pi}\log\frac{b+iz}{b-iz}.$$

Thus
$$X(t)+iY(t) = 2bV/\pi[b^2+z^*(t)^2]. \tag{3}$$

For an electron leaving the filament at an angle θ_0 at time t_0,

$$z(t) = \tfrac{1}{2}ib\exp i\theta_0[1-\exp i(\omega t-\omega t_0)].$$

Let
$$V = A\cos\nu t. \tag{4}$$

Then from (2), the electron is at time t at the point

$$z = \tfrac{1}{2}ib\xi(1-\eta)+bgA\xi I(\xi,\eta), \tag{5}$$

where

$$\xi = \exp i\theta_0, \qquad \eta = \exp i(\omega t-\omega t_0), \qquad g = 8e/m\pi\omega^2 b^2 \tag{6}$$

and
$$I(\xi,\eta) = \int\limits_0^{\omega t-\omega t_0} \frac{\xi\cos(\nu t_0+\nu\phi/\omega)[\eta\exp(-i\phi)-1]i\,d\phi}{[\exp(-i\phi)-1]^2-4\xi^2}.$$

Writing $x = \exp(-i\phi)$, $\cos(\nu t_0+\nu\phi/\omega)$ is a function of x, which we call $F(x)$, and

$$I(\xi,\eta) = \int\limits_1^{1/\eta} \frac{\xi F(x)(-\eta x+1)\,dx}{x(x-1-2\xi)(x-1+2\xi)} \tag{7}$$

taken along an arc of the circle $|x| = 1$ below the real axis.

At a point on the boundary of the cloud of electrons, x and y are unaltered by certain small changes of θ_0 and t_0. Hence the condition to be satisfied on the boundary is $\partial(x,y)/\partial(\theta_0,t_0) = 0$, or

$$\frac{\partial(z,z^*)}{\partial(\xi,\eta)} = 0, \tag{8}$$

in which ξ^*, η^* are to be replaced by ξ^{-1}, η^{-1}. It is easy to see that $\omega t-\omega t_0 = \pi$ in the steady motion, so that $1+\eta$ is small and imaginary. A slight error in η does not affect the radial distance of an electron on the boundary, and it follows from (5) that that distance is given by

$$r/b = 1+gA\times\text{imaginary part of } I(\xi,-1). \tag{9}$$

To calculate $1+\eta$, write I for $I(\xi,\eta)$ and I^{-1} for $I(\xi^{-1},\eta^{-1})$. Then I^*-I^{-1} is an integral round the whole circle $|x| = 1$, so that z^* is expressed easily in terms of ξ and η. The integrand in (7) is rational if ν/ω is an integer, but its residue at the pole $x = 0$ is different in each case. More generally, I is the integral of a rational function if ν/ω is commensurable.

3. Boundary of the cloud of electrons when $\nu = 0$

Here $F(x) = 1$, $I^* = I^{-1}+2i\pi\xi(\xi^2-4)^{-1}$ and (8) becomes, correct to terms of the first order of small quantities,

$$\begin{aligned}1+\eta = igA[I+\xi\partial I/\partial\xi+2\partial I/\partial\eta+I^{-1}+\xi^{-1}\partial I^{-1}/\partial\xi^{-1}+\\ +2\partial I^{-1}/\partial\eta^{-1}+4i\pi\xi^3(\xi^2-4)^{-2}],\end{aligned}$$

where η is put equal to -1 in the small terms after differentiation. The conjugate complex of $I+\xi\partial I/\partial\xi+2\partial I/\partial\eta$ differs from the corresponding -1 function by $4i\pi\xi^3(\xi^2-4)^{-2}$. Write

$$\left.\begin{aligned} J &= \int\limits_1^{-1} \frac{\xi(x+1)\,dx}{x(x-1-2\xi)(x-1+2\xi)},\\ K &= \int\limits_1^{-1} \frac{-\xi[(x+3)(x-1)^2+4\xi^2(x-1)]\,dx}{x(x-1-2\xi)^2(x-1+2\xi)^2}.\end{aligned}\right\} \tag{10}$$

Then $1+\eta = 2igA\times\text{real part of } (-J-K)$.

If we put $J = J_1+iJ_2,...$, where J_1 and J_2 are real functions of ξ, the polar coordinates of the electron on the boundary which left the filament at angle θ_0 are given by

$$r/b = 1+gAJ_2, \qquad \theta = \tfrac{1}{2}\pi+\theta_0+gAK_1. \tag{11}$$

We find

$$\begin{aligned}4(17-8\cos 2\theta_0)J_2 = {} & (-10\sin\theta_0+4\sin 2\theta_0)\log(2\cos\tfrac{1}{2}\theta_0)-\\ & -(10\sin\theta_0+4\sin 2\theta_0)\log(\pm 2\sin\tfrac{1}{2}\theta_0)-\\ & -6\cos\theta_0(\theta_0-2\pi)\pm(-9+3\cos\theta_0+6\cos 2\theta_0)\pi,\end{aligned}$$

$$\begin{aligned}4(17-8\cos 2\theta_0)^2K_1 & \\ = {} & (5-4\cos\theta_0)^2(-10-18\cos\theta_0-8\cos 2\theta_0)\log(2\cos\tfrac{1}{2}\theta_0)+\\ & +(5+4\cos\theta_0)^2(10-18\cos\theta_0+8\cos 2\theta_0)\log(\pm 2\sin\tfrac{1}{2}\theta_0)+\\ & +24(17-8\cos 2\theta_0)\cos\theta_0+(246\sin\theta_0-104\sin 3\theta_0)(\theta_0-2\pi)\pm\\ & \pm(5+4\cos\theta_0)^2(-7\sin\theta_0+4\sin 2\theta_0)\pi,\end{aligned}$$

where $-\pi < \theta_0 < \pi$ and the upper or lower signs are taken according as θ_0 is positive or negative. The functions are tabulated below from 0 to 180°, after which they repeat with reversed sign.

θ_0	J_2	K_1	θ_0	J_2	K_1	θ_0	J_2	K_1
0	1·047	0·590	60°	−0·240	−0·759	120°	−0·689	−0·758
10°	1·054	0·893	70°	−0·369	−0·856	130°	−0·727	−0·718
20°	0·845	0·744	80°	−0·466	−0·880	140°	−0·766	−0·666
30°	0·457	0·269	90°	−0·541	−0·880	150°	−0·798	−0·622
40°	0·161	−0·214	100°	−0·600	−0·851	160°	−0·844	−0·586
50°	−0·068	−0·556	110°	−0·648	−0·820	170°	−0·903	−0·568

It is well known that a magnetron can maintain slow oscillations explained by the characteristic falling over part of its range. The table illustrates this to some extent. Let A be positive, so that the right-hand sector is at the higher potential. Since J_2 is greatest when θ_0 is about 6°, the radial extension of the cloud is greatest near $\theta = 96°$, so that an anode slightly larger than the undisturbed cloud is grazed first at a point on the negative sector. If the anode is the exact size of the undisturbed cloud, an infinitely small potential brings electrons between $\theta_0 = 0$ and 47° to the negative sector and electrons between 227° and 360° to the positive sector, that is more to the positive sector. But since $K_1 = 0{\cdot}59$ when $\theta_0 = 0$, the electrons falling on the negative sector range from $\theta_0 = -0{\cdot}59gA$ to 47° when A is small but finite, so that the range increases with A and the characteristic falls just after its discontinuous rise.

4. Boundary of the cloud when $\nu = \omega$

Here $I(\xi,-1) = -\tfrac{1}{2}[K\exp i\omega t_0 + L\exp(-i\omega t_0)]$, where

$$K = \int_1^{-1} \frac{-\xi(x+1)\,dx}{x^2(x-1-2\xi)(x-1+2\xi)},\quad L = \int_1^{-1} \frac{-\xi(x+1)\,dx}{(x-1-2\xi)(x-1+2\xi)}. \tag{12}$$

The same letter K has been used since the integrals in (10) and (12) are equal. Hence from (9) the radial distance of an electron on the boundary is given by

$$r/b = 1+gA\rho\cos(\omega t_0-\alpha), \tag{13}$$

where $\rho\cos\alpha = -\tfrac{1}{2}(K_2+L_2)$, $\rho\sin\alpha = -\tfrac{1}{2}(K_1-L_1)$. We find

$$\begin{aligned}
&4(17-8\cos 2\theta_0)^2K_2\\
&\quad = (5-4\cos\theta_0)^2(14\sin\theta_0+8\sin 2\theta_0)\log(2\cos\tfrac{1}{2}\theta_0)+\\
&\qquad +(5+4\cos\theta_0)^2(14\sin\theta_0-8\sin 2\theta_0)\log(\pm 2\sin\tfrac{1}{2}\theta_0)-\\
&\qquad -40(17-8\cos 2\theta_0)\sin\theta_0+(106\cos\theta_0-88\cos 3\theta_0)(\theta_0-2\pi)\pm\\
&\qquad \pm(5+4\cos\theta_0)^2(5-9\cos\theta_0+4\cos 2\theta_0)\pi,\\
&4L_1 = -2(1+\cos\theta_0)\log(2\cos\tfrac{1}{2}\theta_0)+\\
&\qquad\qquad +2(1-\cos\theta_0)\log(\pm 2\sin\tfrac{1}{2}\theta_0)-2\sin\theta_0\,\theta_0\pm\sin\theta_0\,\pi,\\
&4L_2 = -2\sin\theta_0\log(2\cos\tfrac{1}{2}\theta_0)-\\
&\qquad\qquad -2\sin\theta_0\log(\pm 2\sin\tfrac{1}{2}\theta_0)+2\cos\theta_0\,\theta_0\pm(1-\cos\theta_0)\pi.
\end{aligned}$$

After 180°, ρ repeats and α is increased by 180°.

θ_0	ρ	α	θ_0	ρ	α	θ_0	ρ	α
0	0·664	−74° 48′	60°	0·378	49° 24′	120°	0·541	119° 30′
10°	0·766	−73° 36′	70°	0·385	72° 18′	130°	0·565	121° 6′
20°	0·746	−65° 6′	80°	0·410	90° 48′	140°	0·581	122°
30°	0·630	−31° 54′	90°	0·448	100° 42′	150°	0·599	121° 48′
40°	0·503	− 6° 0′	100°	0·481	108° 54′	160°	0·617	120° 24′
50°	0·410	21° 48′	110°	0·517	114° 18′	170°	0·641	116° 48′

If the radius of the anode is exactly b, electrons are caught on it when the point θ_0, t_0 lies between the curves $\omega t_0-\alpha = \pm\tfrac{1}{2}\pi$. The range of θ_0 is 180° for most values of t_0, but falls short of it when $15° < \omega t_0 < 28°$ and $195° < \omega t_0 < 208°$, on account of the strong field near the edges of the sectors. It is simpler to ignore the defect and take the range to be 180° throughout. Then half of the electrons emitted between times t_0 and t_0+dt_0 are caught on the anode, which is equivalent to supposing that a thin semicircular sector of positive charge $\tfrac{1}{2}cI\,dt_0$ is emitted by the anode at time $t_0+\pi/\omega$ and moves towards the filament so that it remains nearly semicircular, the ends

moving on arcs of the circle $z = \frac{1}{2}ib\xi(1-\eta)$. The initial semicircle moves round the anode as t_0 increases, θ_0 lying for example between $30°\ 42'$ and $210°\ 42'$ when $\omega t_0 = 60°$ and between $53°$ and $233°$ when $\omega t_0 = 120°$. The aggregate of all the contracting semicircles gives rise to a current as explained in § 5.

5. Calculation of current

Let e be a set of charges inside the magnetron at time t, and let S_1, S_2 be the two sectors of the anode, S_1 extending from polar angles $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$ and S_2 from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. The filament S_0 is supposed to be too small to affect the charges e_1, e_2 induced by e in S_1, S_2 when at some common potential, or the mutual influence of S_1, S_2. Then $e_1+e_2 = -e$, $q_{01} = q_{02}$ and $q_{11} = q_{22}$. The sum of the charges on S_1 and S_2 is $-e-2q_{01}V_0+(q_{11}+q_{12})(V_1+V_2)$ and the difference is

$$e_1-e_2+(q_{11}-q_{12})(V_1-V_2).$$

If V_0 and V_1+V_2 are constant, the sum of the charges is constant, and the rate of increase of the charge on S_2 can be regarded as a current from S_2 to S_1 of amount

$$\frac{K\,dV}{dt}+\frac{de_2}{c\,dt}, \tag{14}$$

where V is the potential of S_2 above S_1 and K the capacity of a condenser whose plates are S_1, S_2.

For a charge e at a complex point z, the image is a charge $-e$ at the inverse point b^2z^{*-1} and the complex potential at a point $b\exp i\theta$ on the anode is

$$V+iU = -2e\log(b\exp i\theta-z)+2e\log(b\exp i\theta-b^2z^{*-1}),$$

where e is reckoned temporarily as a charge per cm. Since $-4i\pi e_2$ is the difference between the values of $V+iU$ when $\theta = \frac{1}{2}\pi$ and $-\frac{1}{2}\pi$,

$$2i\pi e_2/e = -\log(ib-z)+\log(-ib-z)+ +\log(ib-b^2z^{*-1})-\log(-ib-b^2z^{*-1}).$$

It is sufficient to take z from the steady motion, that is $z = \frac{1}{2}ib\xi(1-\eta)$. Since $d\eta/dt = i\omega\eta$,

$$\frac{2\pi\,de_2}{\omega e\,dt}$$

$$= -\frac{\xi\eta}{2+\xi-\xi\eta}-\frac{\xi\eta}{2-\xi+\xi\eta}+\frac{2\xi\eta}{(1-\eta)(1-\eta+2\xi\eta)}+\frac{2\xi\eta}{(1-\eta)(1-\eta-2\xi\eta)}$$

$$= \frac{4\xi^{-1}\eta}{(\eta-1-2\xi^{-1})(\eta-1+2\xi^{-1})}+\frac{4\xi\eta^{-1}}{(\eta^{-1}-1-2\xi)(\eta^{-1}-1+2\xi)}.$$

For a charge e distributed over a semicircle and moving inwards as described in § 4, replace e by the charge $e\,d\theta_0/\pi$ between θ_0 and $\theta_0+d\theta_0$, writing it as $ie\,d\xi^{-1}/\pi\xi^{-1}$ in the first term and $-ie\,d\xi/\pi\xi$ in the second and integrating. Hence the current at time t due to such a semicircle, leaving the filament between polar angles θ_0 and $\theta_0+\pi$ at time t_0, is $R\omega I\,dt_0/\pi^2$, where

$$R = \text{real part of } \frac{i\eta}{\eta-1}\log\left(-\frac{\eta-1-2\xi^{-1}}{\eta-1+2\xi^{-1}}\right)$$

and the correct branch of the logarithm must be taken in each case to represent the continuous passage of θ_0 from one end of its range to the other. The function R is tabulated on p. 248 for 10° intervals of ωt and $\omega t-\omega t_0$, where $\omega t-\omega t_0$ goes from 180° to 360° and R repeats with reversed sign when ωt is increased by 180°. The columns of the table have the same relation to the total current that the ordinates of a curve have to its area, and are treated by Simpson's rule so as to give that current. The result is shown in the following table:

ωt	*Convection current/I*	ωt	*Convection current/I*	ωt	*Convection current/I*
0	−0·222	60°	−0·038	120°	0·198
10°	−0·200	70°	0·015	130°	0·225
20°	−0·176	80°	0·049	140°	0·242
30°	−0·139	90°	0·096	150°	0·248
40°	−0·117	100°	0·129	160°	0·248
50°	−0·068	110°	0·169	170°	0·239

Write $\phi(\omega t)$ for the function just tabulated in the range $0 < \omega t < \pi$. Then if it is expanded in a Fourier's series $\sum (a_n \cos n\omega t + b_n \sin n\omega t)$ in the range $0 < \omega t < 2\pi$, its fundamental is the real part of $(a_1-ib_1)\exp i\omega t$, where

$$a_1-ib_1 = \frac{2}{\pi}\int_0^\pi \phi(\theta)\exp(-i\theta)\,d\theta.$$

We find by Simpson's rule that the fundamental term in the convection current is the real part of

$$(-0{\cdot}233-0{\cdot}098i)I\exp i\omega t. \tag{15}$$

6. Oscillations on Lecher wires

We use the theory (reasonably valid when the frequency is high and the resistance low) which works with self-inductance L, resistance R and capacity C per cm., where R is the resistance per cm. of either

$\omega t - \omega t_0$ \ ωt	0	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	120°	130°	140°	150°	160°	170°
180° +	−0·785	−0·785	−0·785	−0·785	0·785	0·785	0·785	0·785	0·785	0·785	0·785	0·785	0·785	0·785	0·785	0·785	0·785	0·785
190°	−0·789	−0·786	−0·782	−0·771	−0·750	0·844	0·840	0·832	0·827	0·822	0·817	0·815	0·812	0·808	0·805	0·804	0·801	0·797
200°	−0·792	−0·791	−0·780	−0·767	−0·742	−0·701	0·879	0·871	0·861	0·856	0·845	0·839	0·832	0·828	0·819	0·814	0·808	0·801
210°	−0·792	−0·781	−0·769	−0·764	−0·736	−0·700	−0·640	−0·897	0·888	0·878	0·866	0·856	0·846	0·837	0·828	0·818	0·810	0·804
220°	−0·787	−0·771	−0·759	−0·742	−0·716	−0·688	−0·730	−0·534	0·905	0·898	0·881	0·868	0·852	0·840	0·831	0·823	0·807	0·797
230°	−0·770	−0·753	−0·741	−0·719	−0·698	−0·670	−0·720	−0·561	−0·406	0·903	0·891	0·883	0·857	0·843	0·827	0·812	0·798	0·788
240°	−0·740	−0·729	−0·715	−0·690	−0·670	−0·639	−0·610	−0·549	−0·458	−0·242	0·882	0·877	0·858	0·847	0·818	0·800	0·773	0·766
250°	−0·711	−0·690	−0·679	−0·653	−0·628	−0·605	−0·562	−0·520	−0·447	−0·325	−0·011	0·866	0·832	0·848	0·813	0·782	0·761	0·736
260°	−0·688	−0·653	−0·638	−0·605	−0·573	−0·536	−0·508	−0·465	−0·417	−0·317	−0·156	0·267	0·831	0·813	0·789	0·761	0·738	0·718
270°	−0·658	−0·609	−0·580	−0·546	−0·516	−0·479	−0·440	−0·398	−0·346	−0·283	−0·160	0·042	0·581	0·781	0·760	0·729	0·699	0·668
280°	−0·583	−0·567	−0·515	−0·474	−0·439	−0·396	−0·361	−0·314	−0·262	−0·183	−0·127	0·027	0·266	0·760	0·720	0·694	0·671	0·624
290°	−0·520	−0·535	−0·442	−0·411	−0·356	−0·321	−0·260	−0·222	−0·159	−0·101	−0·025	0·063	0·164	0·487	0·900	0·637	0·609	0·571
300°	−0·465	−0·407	−0·360	−0·311	−0·242	−0·189	−0·163	−0·111	−0·062	0·007	0·081	0·164	0·262	0·437	0·673	0·985	0·537	0·547
310°	−0·380	−0·319	−0·271	−0·226	−0·156	−0·104	−0·040	0·009	0·063	0·136	0·195	0·270	0·356	0·446	0·605	0·809	1·021	0·411
320°	−0·282	−0·231	−0·169	−0·128	−0·047	0·003	0·073	0·130	0·169	0·244	0·309	0·380	0·453	0·534	0·622	0·755	0·915	1·031
330°	−1·011	−0·202	−0·117	−0·001	0·064	0·134	0·195	0·255	0·314	0·370	0·433	0·480	0·517	0·617	0·697	0·762	0·872	1·002
340°	−1·111	−0·977	0·060	0·094	0·179	0·239	0·302	0·372	0·400	0·522	0·550	0·592	0·592	0·708	0·755	0·797	0·823	1·015
350°	−0·990	−1·044	−0·927	0·238	0·293	0·328	0·419	0·477	0·530	0·585	0·631	0·683	0·759	0·765	0·808	0·857	0·908	0·943
360° −	−0·984	−1·000	−0·977	−0·838	0·403	0·446	0·510	0·565	0·622	0·672	0·717	0·758	0·799	0·838	0·869	0·903	0·934	0·962

wire at the frequency ν of the oscillations, and $LCc^2 = 1$. Let the wires be joined at $x = 0$ to the sectors of the anode and bridged across at $x = l$, and let the high potential be applied to the anode through large chokes. Then there is a solution in which the current at distance x from the magnetron is

$$[P \exp ikx + Q \exp(-ikx)]\exp i\nu t$$

and the potential

$$(k/\nu C)[P \exp ikx - Q \exp(-ikx)]\exp i\nu t,$$

where $k^2 = \nu^2/c^2 - 2i\nu RC$; or if R is small

$$k = \nu/c - icRC. \tag{16}$$

The potential vanishes when $x = l$, and hence the current at $x = 0$, that is the current through the magnetron, is

$$(i\nu CV/k)\cot kl, \tag{17}$$

where the signs are chosen so that the current flows from S_2 to S_1 and the real parts of the complex quantities are taken in the physical interpretation.

Let the magnetron be working as described in §§ 4 and 5, that is let $\nu = \omega$ and let the radius of the anode be b. Then we have from (4), (14), (15), (16) and (17)

$$[i\nu KA + (-0{\cdot}233 - 0{\cdot}098i)I](\nu - ic^2RC) - \\ - ic\nu CA(\cot \nu l/c + icRCl \operatorname{cosec}^2 \nu l/c) = 0.$$

Neglecting products of R and I and equating real and imaginary parts to zero,

$$A(\nu K - cC \cot \nu l/c) - 0{\cdot}098I = 0,$$

$$c^2RCA(K + Cl \operatorname{cosec}^2 \nu l/c) - 0{\cdot}233I = 0.$$

The zero approximation $R = I = 0$ gives the natural frequencies $\nu \tan \nu l/c = cC/K$. A next approximation gives frequency $\nu + \delta\nu$, where $\delta\nu = 0{\cdot}42R/L$. The amplitude of potential maintained is

$$A = 0{\cdot}233IL/R(K + Cl \operatorname{cosec}^2 \nu l/c). \tag{18}$$

The condition of maintenance, that (18) makes A *positive*, is satisfied.

Take for example a magnetron with anode 4 cm. in diameter, magnetic field 50 gausses, emission 3×10^{-6} amperes, attached to Lecher wires of copper, each 1 mm. in diameter and 2 cm. apart. The capacity of the magnetron itself is taken as 1 cm. Then

$\nu = \omega = 8{\cdot}9\times 10^8$, the resistance of each wire at that frequency is 0·009 ohms per cm., the steady anode potential 225 volts, $l = 39$ cm. for the lowest mode, $A = 0{\cdot}246$ volts and $\delta\nu/\nu = 1/3500$.

7. Effect of the filament on returning electrons

We shall discuss the case $\nu = \omega$, for which

$$I(\xi,\eta) = \frac{\exp i\omega t_0}{2}\int\limits_1^{1/\eta} \frac{\xi(-\eta x+1)\,dx}{x^2(x-1-2\xi)(x-1+2\xi)}+$$

$$+\frac{\exp(-i\omega t_0)}{2}\int\limits_1^{1/\eta} \frac{\xi(-\eta x+1)\,dx}{(x-1-2\xi)(x-1+2\xi)}.$$

If $\omega t-\omega t_0 = 2\pi$, integration is round the whole circle. Thus the electron would be at the point

$$z = -i\pi bgA\xi^2(1+4\xi^2)(1-4\xi^2)^{-2}\exp i\omega t_0$$

at time $t_0+2\pi/\omega$, if it were not for the repulsion of the filament, and would have a complex velocity w nearly equal to $\frac{1}{2}b\omega\xi$. This electron approaches the filament with a moment of velocity

$$\begin{aligned} h &= \tfrac{1}{2}i(zw^*-z^*w) \\ &= \tfrac{1}{2}\pi b^2 g\omega A\times \text{real part of } \xi(1+4\xi^2)(1-4\xi^2)^{-2}\exp i\omega t_0 \end{aligned}$$

round it. The greatest numerical value of h, considered as a function of t_0, is

$$\tfrac{1}{2}\pi b^2 g\omega A\,|\xi(1+4\xi^2)/(1-4\xi^2)^2| = 8eA(17+8\cos 2\theta_0)^{\frac{1}{2}}/m\omega(17-8\cos 2\theta_0),$$

so that the greatest value of h for any returning electron is $20eA/9m\omega$. If $A = 0$, finite velocity of emission will make some of the electrons (those, at least, which leave the filament normally) penetrate it and be caught on return. It remains to find whether, with any given amplitude of oscillating potential, most of the electrons are caught or not. The theory is simplified by assuming normal emission with a fixed velocity, corresponding to potential B.

Let an electron at a large distance from the filament be moving directly towards it with such a velocity that its kinetic energy on arriving at the filament $r = a$ is eB, and let it be displaced laterally so that the moment of its velocity becomes h. Then

$$\tfrac{1}{2}m(\dot r^2+r^2\dot\theta^2)-eV_0\log r/a \div \log b/a = \text{const.},$$

where $r^2\dot\theta = h$ and the constant is the same as for the undisplaced

electron, that is eB. If the displaced electron just reaches the filament before turning, $mh^2/2a^2 = eB$. Since the high potential V_0 is approximately equal to $m\omega^2 b^2/8e$, the least radius of filament which will catch all the returning electrons is given by

$$81a^2 V_0 B = 25b^2 A^2. \tag{19}$$

If $V_0 = 225$, $A = 0{\cdot}246$ and $B = 0{\cdot}1$, $a/b = 1/38$. This ratio is too large for the paths to be accurate circles, so that the theory is only qualitative at the best.

EXPANSIONS BY THE FRACTIONAL CALCULUS

By W. FABIAN (*London*)

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1. Introduction

Bernoulli* defines $\left(\frac{d}{dx}\right)^{-\lambda}f(x)$ for real values of λ by the series

$$\left(\frac{d}{dx}\right)^{-\lambda}f(x) = \frac{1}{\Gamma(\lambda)}\sum_{n=0}^{\infty}(-1)^n\frac{f^{(n)}(x)}{n!\,(\lambda+n)}x^{\lambda+n}. \tag{1}$$

Bourlet† and Davis‡ have shown that this series can be obtained from Riemann's definition||

$$\left(\frac{d}{dx}\right)^{-\lambda}f(x) = \frac{1}{\Gamma(\lambda+\gamma)}\left(\frac{d}{dx}\right)^{\gamma}\int_0^x (x-t)^{\lambda+\gamma-1}f(t)\,dt,$$

where γ is the least non-negative integer such that $\lambda+\gamma > 0$. With Riemann's definition, which I have adopted in my papers on the Fractional Calculus,** I have shown that, under certain conditions,

$$\left(\frac{d}{dx}\right)^{-\lambda}\left(\frac{d}{dx}\right)^{\lambda}f(x) = f(x).$$

This suggests that an expansion of $f(x)$ can be obtained under appropriate conditions by applying Bernoulli's formula to $\left(\frac{d}{dx}\right)f(x)$. Suitable conditions, under which a function can be expanded in this manner, will be determined in this paper.

2. Fractional Integration

A λth integral of $f(z)$ along a simple curve l is defined by

$$D^{-\lambda}(l_a)f(z) = \frac{D^{\gamma}}{\Gamma(\lambda+\gamma)}\int_a^z (z-t)^{\lambda+\gamma-1}f(t)\,dt,$$

* Pincherle, *Mem. d. R. Acad. di Bologna* (*Sci.*) (5) 9 (1902), 745, gives Bernoulli's form of the fractional differential coefficient.

† Bourlet, *Ann. de l'École Normale* (3) 14 (1897), 154.

‡ Davis, *American J. of Math.* 46 (1924), 95–109.

|| Riemann, *Ges. Werke* (1876), xix. 331–44.

** Fabian, *Phil. Mag.* (7) 20 (1935), 781–9; 21 (1936), 274–80. *American J. of Math. and Phys.* 15 (1936). *Math. Gaz.* 20 (1936), 88–92; 249–253.

Expansion (1) is established here for the complex variable and complex values of λ.

where γ is the least non-negative integer such that $R(\lambda)+\gamma > 0$; D stands for d/dz, and the integration and differentiation are along l.

I have shown elsewhere* that, if $f(z)$ is analytic throughout a bounded simply-connected region E, which contains l, then each branch of $D^{-\lambda}(l_a)f(z)$ is analytic inside E, except possibly at a. Under these conditions the value of each branch of $D^{-\lambda}(l_a)f(z)$, given a and z, is independent of l.

3. Expansions

THEOREM 1. *Let $f(z)$ be analytic within a circle of centre z_0 which contains l in its interior. Let λ be such that $R(\lambda) < 0$, and let*

$$f(a) = f'(a) = \ldots = f^{(\gamma-2)}(a) = 0 \quad \textit{if} \quad R(\lambda) \leqslant -1.$$

Then
$$f(z_0) = \frac{1}{\Gamma(-\lambda)} \sum_{n=0}^{\infty} (-1)^n \frac{f_{\lambda-n}(z_0)}{n!\,(n-\lambda)} (z_0-a)^{n-\lambda},$$

provided† that the Taylor series for $f_\lambda(z)$ at z_0 converges on l uniformly to $f_\lambda(z)$.

Proof. By the properties of $f_\lambda(z)$ given in § 2, the circle of convergence of the Taylor series for $f_\lambda(z)$ at z_0 extends at least as far as a. Expanding therefore $f_\lambda(t)$ at $t = z_0$ in a Taylor series, we have, since, by hypothesis, this series converges to $f_\lambda(t)$ uniformly on l up to a,

$$D^\lambda(l_a)f_\lambda(z_0) = \frac{1}{\Gamma(-\lambda)} \int_a^{z_0} (z_0-t)^{-\lambda-1} \left\{ \sum_{n=0}^{\infty} \frac{f_{\lambda-n}(z_0)}{n!} (t-z_0)^n \right\} dt$$

$$= \frac{1}{\Gamma(-\lambda)} \sum_{n=0}^{\infty} (-1)^n \frac{f_{\lambda-n}(z_0)}{n!\,(n-\lambda)} (z_0-a)^{n-\lambda}.$$

By a previous theorem,‡

$$D^\lambda(l_a)f_\lambda(z_0) = f(z_0).$$

The proof of the theorem is completed.

The expansion just obtained can also be proved to hold under different hypotheses. These are given in Theorem 3, for which the

* *Math. Gaz.* 20 (1936), 249.

† $f_{\lambda-n}(z_0)$ stands for $D^{n-\lambda}(l_a)f(z_0)$, and $f_\lambda(z)$ stands for $D^{-\lambda}(l_a)f(z)$. This notation will be used, when no ambiguity can arise.

‡ Fabian, *Phil. Mag.* (7) 20 (1935), 781.
The conditions $f(a) = f'(a) = \ldots = f^{(\gamma-2)}(a) = 0$ when $R(\lambda) \leqslant -1$ are required for this step of the proof.

following preliminary theorem is required:

THEOREM 2. *Let $f(z)$ be analytic on l. Let λ be any number, and let*

$$f(a) = f'(a) = \ldots = f^{(\gamma-2)}(a) = 0 \quad \text{if} \quad R(\lambda) \leqslant -1.$$

Then, if δ is any non-negative integer such that $R(-\lambda)+\delta > 0$,

$$f(z) = \frac{1}{\Gamma(-\lambda)} \sum_{m=0}^{n-1} (-1)^m \frac{f_{\lambda-m}(z)}{m!\,(m-\lambda)} (z-a)^{m-\lambda} +$$

$$+\frac{(-1)^{n-1}}{(n-1)!} \sum_{m=0}^{\delta-1} \sum_{s=0}^{m} \frac{m!\,\Gamma(m-\lambda+n)(z-a)^{n+s-\lambda}}{s!\,(m-s)!\,\Gamma(m-\lambda+1)\Gamma(n+s-\lambda+1)} f_{\lambda-n-s}(z)+$$

$$+\frac{D^\delta}{\Gamma(\delta-\lambda)} \int_a^z \left\{(z-t)^{\delta-\lambda-1} \int_z^t f_{\lambda-n}(t)\, dt^n\right\} dt$$

for all positive integers n; all integrations are along l, and the sum $\sum_{m=0}^{\delta-1} \sum_{s=0}^{m}$ is omitted if $\delta = 0$.

Proof. $$f(z) = D^\delta D^{\lambda-\delta}(l_a) f_\lambda(z),$$

by a previous theorem,*

$$= \frac{D^\delta}{\Gamma(\delta-\lambda)} \int_a^z (z-t)^{\delta-\lambda-1} \left\{\frac{d}{dt} \int_z^t f_\lambda(\tau)\, d\tau\right\} dt$$

$$= \frac{D^\delta}{\Gamma(\delta-\lambda)} \int_a^z (z-t)^{\delta-\lambda-1}\, dt \left\{\sum_{m=0}^{n-1} \frac{f_{\lambda-m}(z)}{m!} (t-z)^m + \int_z^t f_{\lambda-n}(\tau)\, d\tau^n\right\}$$

(on integrating $\int_z^t f_\lambda(\tau)\, d\tau$ by parts n times)

$$= \frac{D^\delta}{\Gamma(\delta-\lambda)} \sum_{m=0}^{n-1} (-1)^m \frac{f_{\lambda-m}(z)}{m!\,(m+\delta-\lambda)} (z-a)^{m+\delta-\lambda} +$$

$$+\frac{D^\delta}{\Gamma(\delta-\lambda)} \int_a^z \left\{(z-t)^{\delta-\lambda-1} \int_z^t f_{\lambda-n}(t)\, dt^n\right\} dt,$$

which comes down to the form enunciated in the theorem, when the differentiations of the first n terms of this expansion are performed.

* Fabian, *Phil. Mag.* (7) 20 (1935), 781.

From the theorem just proved we can immediately deduce the following theorem, which gives the expansion in Theorem 1:

THEOREM 3. *Let $f(z)$ be analytic on l. Let λ be any number, and let*

$$f(a) = f'(a) = \ldots = f^{(\gamma-2)}(a) = 0 \quad \textit{if} \quad R(\lambda) \leqslant -1.$$

Then, for those values of z for which

$$\frac{D^{\delta}}{\Gamma(\delta-\lambda)} \int\limits_{a}^{z} \left\{ (z-t)^{\delta-\lambda-1} \int\limits_{z}^{t} f_{\lambda-n}(t)\, dt^{n} \right\} dt$$

tends to zero, and $(z-a)^{n-\lambda} f_{\lambda-n}(z)$ remains bounded when the positive integer n tends to infinity,

$$f(z) = \frac{1}{\Gamma(-\lambda)} \sum_{n=0}^{\infty} (-1)^{n} \frac{f_{\lambda-n}(z)}{n!\,(n-\lambda)} (z-a)^{n-\lambda};$$

all integrations being along l, and δ being a non-negative integer such that $R(-\lambda)+\delta > 0$.

CONJUGATE MATRICES

By A. R. RICHARDSON (*Swansea*)

[Received 29 May 1936]

1. CONJUGATE matrices* are usually defined as a set satisfying the conditions:

(i) they are commutative;

(ii) their symmetric functions are scalar matrices which are equal to the corresponding symmetric functions of the scalar roots of the characteristic equation of one of the matrices;

(iii) all the matrices have the same *characteristic* equation;

(iv) all are of a specified order n.

P. Franklin† has generalized this by omitting condition (iii), whilst Sokolnikoff‡ replaces (iii) by the condition that the matrices have the same *reduced* equation. Condition (iv) is not mentioned as such but is always implied. In some respects this theory resembles that of algebraic numbers but the analogy is very incomplete.§ For example, condition (iv) requires that the matrices involve the latent roots of the characteristic equation and also the nth roots of unity. In this note we show that, if in place of condition (iv) we require the matrices to be of the same order as the Galois group of the reduced equation, then there is a complete simple isomorphism between the fields of the original equation and that of its matrix roots. Also, if the elements of the matrix roots are to belong to the field of the coefficients of the equation, then the order of the complete matrix-representation is the smallest possible. If the roots are represented by matrices of order higher than that of the Galois group, then the realm formed by adjoining them to the rational field may contain divisors of zero. Purely rational operations suffice to enable us to calculate the conjugate roots and the automorphisms of the Galois field.

2. If the order of the matrix solutions of an equation is restricted to be that of the matrix coefficients, it appears that certain equations

* Pp. 24–5. All references, unless otherwise stated, are to *The Theory of Matrices*, C. C. MacDuffee (J. Springer, Berlin 1935).

† P. Franklin, pp. 24–5.

‡ E. S. Sokolnikoff, *American J. of Math.* 55 (1933), 167.

§ A. A. Bennett, p. 24.

will have no solution, others a restricted number. For example, P. Franklin* shows that

$$x^2 = \begin{bmatrix} 0, & -1 \\ 0, & 0 \end{bmatrix} \tag{1}$$

has no solution in matrices of order 2. Also it may be verified that

$$x^3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{2}$$

has only one independent rational matrix solution of order 3. Normality is restored if the order of the matrix solution is raised. Thus (1) has the two rational conjugate solutions

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix};$$

whilst (2) has the three rational conjugate solutions

$$\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & -2 & 0 & -2 & 0 \\ -1 & 0 & 0 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -2 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

Evidently the number of solutions is never reduced when the matrix order is raised. We proceed to prove that, for a given equation, there is a lowest order to the matrices of a complete set of rational conjugates.

There exists a complete set of n rational conjugate matrix solutions, each of order $n!$, of the equation

$$f_n(x) = x^n - a_1 x^{n-1} + a_2 x^{n-2} + \ldots + (-1)^n a_n = 0 \tag{3}$$

over the field κ of characteristic 0.

* p. 25 (2).

We assume throughout that $f_n(x)$ has no repeated zeros. Denote the complex zeros by $\alpha_1, \alpha_2, \ldots, \alpha_n$, and the corresponding matrix zeros by $A_1, A_2, \ldots, A_n$. One root of (3) is

$$A_1 = \begin{bmatrix} a_1 & -a_2 & a_3 & . & . & (-1)^{n-1}a_n \\ 1 & 0 & 0 & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}. \tag{4}$$

Next

$$f_n(x)/(x-A_1) = x^{n-1}-p_1x^{n-2}+p_2x^{n-3}+\ldots+(-1)^{n-1}p_{n-1} = 0,$$

where $p_s = a_s-a_{s-1}A_1+a_{s-2}A_1^2+\ldots+(-1)^sA_1^s$, is an equation with commutative matrix coefficients each of order n and has one rational matrix solution of order $n(n-1)$, viz.:

$$A_2 = \begin{bmatrix} p_1 & -p_2 & p_3 & . & . & (-1)^{n-2}p_{n-1} \\ 1 & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & 0 \end{bmatrix}. \tag{5}$$

Evidently A_2 is commutative with A_1 regarded as being of order $n(n-1)$ by repeating* A_1 $(n-1)$ times in the leading diagonal.

Hence

$$f_n(x)/(x-A_1)(x-A_2)$$
$$= x^{n-2}-q_1x^{n-3}+q_2x^{n-4}+\ldots+(-1)^{n-2}q_{n-2} = 0 \tag{6}$$

has commutative coefficients each of order $n(n-1)$ which are symmetric polynomials in A_1 and A_2. Proceeding in this way we reach a complete set of n rational matrices each of order $n!$ which

(i) satisfy the same reduced equation $f_n(x) = 0$ and are therefore all derogatory,

(ii) are mutually commutative.

Moreover, in the repeated factorization of $f_n(x)$ we may replace the matrices at each step by the original complex zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$ and, since multiplication is commutative,

(iii) the elementary symmetric functions of the matrix roots $A_1, \ldots, A_n$ are the same as the corresponding elementary symmetric functions of the complex roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Hence $A_1, A_2, \ldots, A_n$ form a complete set of rational conjugate matrices† each of order

* We shall often use the same symbol to denote matrices of different orders.

† i.e. they satisfy conditions (i) and (ii).

$n!$ which are solutions of (3). Even if $f_n(x)$ is reducible or has repeated zeros, the result holds provided that the coefficients are commutative. If the coefficients are matrices of order m, then our solutions are matrices of order $n!m$.

Certain equations have a complete set of rational conjugate solutions in matrices of order less than $n!$. Thus

$$x^3+x^2-2x-1 = 0 \tag{7}$$

has not only

$$A_1 = \begin{bmatrix} -1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -2 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ -1 & 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}, \tag{8}$$

obtained by the preceding method, but also the complete set of conjugates

$$A_1 = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}. \tag{9}$$

In both (8) and (9) the elementary symmetric functions of the roots are also those of (7) but there is an important distinction; in (9) $A_2 = A_1^2-2$, $A_3 = 1-A_1^2-A_1$, but these relations no longer hold for the solutions (8). These relations are replaced by

$$(A_2-A_1^2+2)(A_3+A_1^2+A_1-1) = 0,$$

of which each factor is a divisor of zero. Next,

The necessary and sufficient conditions that an equation of degree n shall be normal is that it shall have a complete set of non-derogatory conjugate matrix solutions each of order n.*

* i.e. a Galois equation.

If $f_n(x) = 0$ is a normal equation, every complex root is a polynomial in any one, i.e.

$$f_n(x) = (x-\alpha)\{x-r_2(\alpha)\}...\{x-r_n(\alpha)\} = 0 \tag{10}$$

where $r_i(\alpha)$ is a rational polynomial in the complex root α. We may identify this form for $f_n(x)$ with the form (3) by means of rational operations and the fact that $f_n(x) = 0$. Hence, since there is always at least one matrix root A_1 of order n, we may replace α by A_1 in the argument and

$$f_n(x) = (x-A_1)\{x-r_2(A_1)\}...\{x-r_n(A_1)\} = 0, \tag{11}$$

so that $A_1, r_2(A_1),..., r_n(A_1)$ are matrix roots which form a complete conjugate system.

Conversely, if all the matrix roots are of degree n and if A_1 is non-derogatory,* any matrix commutative with A_1 is a polynomial in A_1. Since the conjugates must be commutative with A_1 it follows that they are polynomials in A_1, and $f_n(x)$ may be written in the form (11), which, by a repetition of the first part of the argument, may be identified with (10). Hence $f_n(x) = 0$ is normal. We see that Sylvester's theorem that, if A is non-derogatory,† all matrices commutative with A are polynomials in A and the theorem that in a normal equation all the roots are polynomials in any one of them are merely different ways of stating the same fact.

If a polynomial relation $\phi(\alpha_1,..., \alpha_n) = 0$ connects the complex roots of $f_n(x) = 0$, then some $\phi(A_1, A_2,..., A_n)$ is either zero or a divisor of zero in any matrix representation of the roots.

For $\psi = \prod_s \phi(\alpha_1,..., \alpha_n)$, defined as the product of all the polynomials obtained from ϕ by operations of the symmetric group in the suffixes, is zero. Hence, since $A_1,..., A_n$ is a complete set of conjugates,

$$\prod_s \phi(A_1,..., A_n) = 0.$$

Therefore some factor $\phi(A_1,..., A_n)$ is either zero or a divisor of zero.

If in any matrix representation $\phi(A_1,..., A_n) = 0$, then

$$\phi(\alpha_1,..., \alpha_n) = 0.$$

* p. 94. The matrix A_1 of order n of § 2 is non-derogatory and can be transformed into the A_1 above.

† A may be non-derogatory even though $f_n(x)$ is reducible. P. Franklin, Ex. 3, shows that $x^2 = 1$ has the complete sets of roots

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}; \quad \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}:$$

the members of the first set are derogatory whilst those of the second set are not.

For $\prod_s \phi(A_1,\ldots,A_n) = 0$ is symmetric in the roots and is therefore unchanged when we replace $A_1,\ldots,A_n$ by $\alpha_1,\ldots,\alpha_n$. Thus $\prod_s \phi(\alpha_1,\ldots,\alpha_n) = 0$. Hence one of the factors is zero, say $\phi(\beta_1,\ldots,\beta_n) = 0$, the β's being some permutation of $\alpha_1,\ldots,\alpha_n$. As it is rational, we must also have

$$\phi(\alpha_1,\ldots,\alpha_n) = 0.$$

If A be a matrix of order n satisfying an irreducible normal equation $f_n(x) = 0$ of degree n, then there is no polynomial in A which is a divisor of zero.

For the latent roots of any polynomial $g(A)$ are $g(\alpha_1),\ldots,g(\alpha_n)$. Hence by a transformation, possibly irrational,

$$g(A) = \begin{bmatrix} g(\alpha_1) & & & \\ & g(\alpha_2) & & \\ & & \ddots & \\ & & & g(\alpha_n) \end{bmatrix}.$$

If this is zero, then every $g(\alpha_s) = 0$ $(s = 1\ldots n)$, and, since $f_n(x)$ is irreducible, $f_n(x)/g_n(x)$ and therefore $g(A)$ is zero and not merely a divisor of zero. It follows that

The realm $\kappa(A)$ obtained by adjoining the matrix A of order n, solution of a normal equation of degree n, to the field κ is a field which is simply isomorphic with the field $\kappa(\alpha)$.

This is an extension of a theorem of Pierce* in which he proves a corresponding theorem for cyclic equations.

If the representation of the roots of a normal equation of degree n is of order a multiple of n, then the realm $\kappa(A_1,\ldots,A_n)$ may contain divisors of zero.

For A_1 can be transformed into a direct sum of matrices each of which is of order n and satisfies $f_n(x) = 0$. Hence, if $A_r = \beta_r(A_1)$, A_r will be similarly expressed as a direct sum of matrices of order n. Such representations are therefore mere duplications, in the leading diagonal, of the normal representation and will not be regarded as of true order exceeding n. If the order is a true multiple of n, then $\prod \{A_r - \beta_r(A_1)\}$ is symmetric and is rational and therefore zero. Hence $A_r - \beta_r(A_1)$ may be a divisor of zero.

3. The resolvent matrix

The general number in the realm $\kappa(A_1, A_2,\ldots,A_n)$, obtained by adjoining the conjugate roots $A_1, A_2,\ldots,A_n$ of § 2 to the field of the

* p. 25.

coefficients, κ, is

$$V_1 = \sum_{i=1}^{n} x_i A_i, \tag{12}$$

where x_i is an indeterminate. There are $n!-1$ associates of V_1 which are formed from it by permuting the A's by means of every possible permutation of the suffixes. It will be proved in § 4 that matrices T_{ij} exist such that

$$T_{ij}^{-1} A_i T_{ij} = A_j, \qquad T_{ij}^{-1} A_j T_{ij} = A_i,$$
$$T_{ij}^{-1} A_p T_{ij} = A_p \quad (p \neq i, j).$$

Hence the associates of V_1 being obtained from it by transformation satisfy the same characteristic and reduced equations which are therefore of degree $n!$, *and V_1 is non-derogatory* in the case now considered. Further, the A's are commutative and so are the V's. Hence

The characteristic equation for the resolvent V_1 is normal, and V_1 and its associates form a complete set of rational conjugate matrices.

It follows that the characteristic equation for V_1 is also a complete Galois resolvent for $f_n(x) = 0$. A further consequence is that V_i and A_i are all rational polynomials in V_1. There are two cases to consider:

(i) if the characteristic equation of V_1 is *irreducible*

$$\kappa(A_1, A_2, \ldots, A_n) = \kappa(V_1)$$

is a field, and it is impossible to transform all the A's simultaneously into direct sums of matrices of order less than $n!$;

(ii) if the characteristic equation is *reducible*, a rational matrix T can be found such that $T^{-1}V_1 T$ is the direct sum of non-derogatory matrices of order less than $n!$, the characteristic equations of which are the irreducible factors of the characteristic equation of V_1.

Also, since every V_i is a rational polynomial in V_1, each V_1, and therefore every A_i, is simultaneously transformed by T into a direct sum of matrices whose orders are the same as the degrees of the irreducible factors of the characteristic polynomial of V_1, that is of the complete Galois resolvent. Further, since every transform of a zero is also a zero and since

$$A_i = A_{i1} \dot{+} A_{i2} \dot{+} \ldots \dot{+} A_{ir},$$

then $$f(A_i) = f(A_{i1}) \dot{+} f(A_{i2}) \dot{+} \ldots \dot{+} f(A_{ir}) = 0.$$

Hence $f(A_{ip}) = 0$ $(p = 1, \ldots, r)$. Also, since the A's are commutative, so are the A_i's, and the set A_i $(i = 1, \ldots, n)$ forms a complete set of

matrix solutions of $f_n(x) = 0$. As this is supposed irreducible and as the transform of a conjugate set is also a conjugate set, the set A_i $(i = 1,..., n)$ may be taken to be the same matrices arranged in a different order. Evidently the complete resolvent splits into r irreducible conjugate factors each of degree $s = n!/r$, but since the characteristic equation for V_{ip} is different from that for V_{iq} and each is irreducible, it is impossible to transform $V_{ip} \to V_{iq}$. Hence the effect of transforming by T is to separate the $n!$ conjugate matrices V_i into classes in which the leading elements of each class form a set of conjugate solutions of a Galois resolvent. In Ex. 8

$$V_1 = A_1+2A_2+3A_3 = \begin{bmatrix} -1 & -2 & -1 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 0 & 1 \\ 0 & -1 & -2 & 1 & 1 & -2 \\ -1 & 0 & 0 & -1 & -4 & -2 \\ 0 & -1 & 0 & -2 & -3 & 0 \\ 0 & 0 & -1 & 0 & -2 & -3 \end{bmatrix}.$$

The complete resolvent is

$$(x^3+6x^2+5x+1)(x^3+6x^2+5x-13) = 0,$$

which is reducible although V_1 is non-derogatory. The other roots are

$$V_2 = 2A_1+3A_2+A_3, \qquad V_3 = 3A_1+A_2+2A_3, \tag{13}$$

$$V_4 = A_1+3A_2+2A_3, \quad V_5 = 2A_1+A_2+3A_3, \quad V_6 = 3A_1+2A_2+A_3. \tag{14}$$

T, T^{-1} are respectively

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 1 & -4 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & -1 \\ -1 & -1 & 2 & -1 & 0 & 1 \end{bmatrix}, \qquad \frac{1}{7}\begin{bmatrix} 4 & -9 & -5 & -1 & -10 & -4 \\ -5 & -1 & 1 & -4 & -5 & -2 \\ 1 & -4 & -3 & -2 & -6 & -1 \\ 4 & 5 & 2 & -1 & 4 & 3 \\ 2 & 6 & 1 & 3 & 2 & -2 \\ 1 & 3 & 4 & -2 & 1 & 6 \end{bmatrix}.$$

These transform V_1 and its conjugates into $T^{-1}V_1T$, $T^{-1}V_2T$, $T^{-1}V_3T$, $T^{-1}V_4T$, $T^{-1}V_5T$, $T^{-1}V_6T$, which are respectively equal to

$$\left[\begin{array}{ccc:ccc} -2 & -3 & -1 & & & \\ -1 & -3 & -1 & & & \\ -1 & -2 & -1 & & & \\ \hdashline & & & 0 & -3 & -2 \\ & & & -2 & -2 & 1 \\ & & & 1 & -1 & -4 \end{array}\right], \qquad \left[\begin{array}{ccc:ccc} 0 & 0 & -1 & & & \\ -1 & -1 & 2 & & & \\ 2 & 1 & -5 & & & \\ \hdashline & & & -4 & 0 & 1 \\ & & & 1 & -3 & -2 \\ & & & -2 & -1 & 1 \end{array}\right],$$

$$\left[\begin{array}{ccc:ccc} -4 & 3 & 2 & & & \\ 2 & -2 & -1 & & & \\ -1 & 1 & 0 & & & \\ \hdashline & & & -2 & 3 & 1 \\ & & & 1 & -1 & 1 \\ & & & 1 & 2 & -3 \end{array}\right], \quad \left[\begin{array}{ccc:ccc} 0 & -3 & -2 & & & \\ -2 & -2 & 1 & & & \\ 1 & -1 & -4 & & & \\ \hdashline & & & -2 & -3 & -1 \\ & & & -1 & -3 & -1 \\ & & & -1 & -2 & -1 \end{array}\right],$$

$$\left[\begin{array}{ccc:ccc} -4 & 0 & 1 & & & \\ 1 & -3 & -2 & & & \\ -2 & -1 & 1 & & & \\ \hdashline & & & 0 & 0 & -1 \\ & & & -1 & -1 & 2 \\ & & & 2 & 1 & -5 \end{array}\right], \quad \left[\begin{array}{ccc:ccc} -2 & 3 & 1 & & & \\ 1 & -1 & 1 & & & \\ 1 & 2 & -3 & & & \\ \hdashline & & & -4 & 3 & 2 \\ & & & 2 & -2 & -1 \\ & & & -1 & 1 & 0 \end{array}\right].$$

The leading elements in the transforms of V_1, V_2, V_3 all satisfy the resolvent $x^3+6x^2+5x+1 = 0$ whilst the lower elements satisfy the other factor of the complete resolvent, i.e. $x^3+6x^2+5x-13 = 0$. The effect of transforming A_1, A_2, A_3 by T is

$$T^{-1}A_1 T = A_1\dot{+}A_1, \qquad T^{-1}A_2 T = A_2\dot{+}A_3, \qquad T^{-1}A_3 T = A_3\dot{+}A_2,$$

where on the right-hand side we refer to the matrices (8) in their order along the leading diagonal from left to right.*

It follows from this that there is a least value to the order of a complete conjugate rational matrix representation of the roots of an irreducible equation and that this order is a divisor of $n!$ For this representation $\kappa(V_1)$ is a field, whereas for true† representations by matrices of higher order, e.g. $n!$, $\kappa(V_1)$ is not a field but contains divisors of zero as we have noted. This fact is sometimes useful. In (8) and in the field $\kappa(A_1)$

$$V_1 = \begin{bmatrix} -2-A & A^2+A-2 \\ -1 & -3-2A \end{bmatrix}$$

with resolvent

$$\{V-(A+1)^2\}\{V+A^2-A-4\} = 0.$$

Since V is non-derogatory it follows that $\kappa(V_1)$ in matrices of order

* P. Franklin's Ex. 3 (p. 25) is explained by the fact that, when the characteristic equation is reducible, solutions may be constructed for which V is expressed as a direct sum of elements some of which satisfy the same reduced equation and not, as with our V_1, the conjugate resolvents. Such solutions will be derogatory.

† i.e. not mere repetitions of representations of lower order.

six contains divisors of zero and the order of the representation is unnecessarily high.

4. Transforming matrices and the Galois group

The transform of any matrix root is also a root and the transform of a complete set of conjugates is also a complete set of conjugates. Also* any two roots of the same reduced irreducible equation can be transformed into each other. We proceed to show that for the representation of order $n!$ of the conjugate sets there exist rational matrices which interchange any pair of roots whilst leaving the others unaltered. The proof is written out for the case $n = 5$ but is quite general.

There is a matrix of order $(n-1)$, having as elements matrices of order n, which interchanges A_1 and A_2 (of § 2).

Write

$$A_1 = \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{bmatrix}, \qquad A_2 = \begin{bmatrix} p_1 & -p_2 & p_3 & -p_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where $p_s = a_s - a_{s-1}A + a_{s-2}A^2 + \ldots + (-1)^s A^s \quad (s = 1,\ldots, 4), \qquad (15)$

and let

$$P = \begin{bmatrix} \delta_1 A^3 & \delta_2 A^3 & \delta_3 A^3 & \delta_4 A^3 \\ \delta_1 A^2 & \delta_2 A^2 & \delta_3 A^2 & \delta_4 A^2 \\ \delta_1 A & \delta_2 A & \delta_3 A & \delta_4 A \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{bmatrix},$$

where δ_s is a matrix of order $n = 5$.

We require that $PA_1 = A_2 P$ and $PA_2 = A_1 P$.

Now $PA_1 = A_2 P$ requires that

$$\delta_1 A^4 = p_1 \delta_1 A^3 - p_2 \delta_1 A^2 + p_3 \delta_1 A - p_4 \delta_1 \tag{16}$$†

which, using (15), may also be written

$$A^4 \delta_1 = A^3 \delta_1 p_1 - A^2 \delta_1 p_2 + A \delta_1 p_3 - \delta_1 p_4. \tag{17}$$

Evidently (16) is also satisfied by $\delta_1 A$, $\delta_1 A^2$, $\delta_1 A^3$. Further δ_2, δ_3, δ_4 must also satisfy (16). Again $PA_2 = A_1 P$ requires that

$$\delta_1 p_1 + \delta_2 = A\delta_1, \qquad -\delta_1 p_2 + \delta_3 = A\delta_2, \qquad \delta_1 p_3 + \delta_4 = A\delta_3,$$
$$-\delta_1 p_4 = A\delta_4. \tag{18}$$

* Wedderburn, *Lectures on Matrices*, p. 114.

† The solution of (16) involves only rational operations.

For these to be consistent

$$-\delta_1 p_4 = A\delta_4 = -A\delta_1 p_3 + A^2\delta_3 = -A\delta_1 p_3 + A^2\delta_1 p_2 + A^3\delta_2$$
$$= -A\delta_1 p_3 + A^2\delta_1 p_2 - A^3\delta_1 p_1 + A^4\delta_1.$$

Hence, if δ_1 be chosen to satisfy (16) and δ_2, δ_3, δ_4 be calculated from (18) in terms of δ_1, the P will interchange A_1 and A_2.

Since the equations found by multiplying on the right by powers of A are left linearly dependent, the equation

$$p_4 x - p_3 xA + p_2 xA^2 - p_1 xA^3 + xA^4 = 0, \tag{19}$$

is singular.*

A particular solution is easily found if we use the fact that

$$\phi^{-1}A\phi = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_5 \end{pmatrix}$$

where

$$\phi = \begin{bmatrix} \alpha_1^4 & \alpha_2^4 & . & . & \alpha_5^4 \\ \alpha_1^3 & \alpha_2^3 & . & . & \alpha_5^3 \\ . & . & . & . & . \\ \alpha_1 & \alpha_2 & . & . & \alpha_5 \\ 1 & 1 & . & . & 1 \end{bmatrix},$$

and it is easily verified that such a particular solution of (19) is

$$\delta_1 = x = \begin{bmatrix} -1 & 0 & 0 & 0 & s_4 \\ 0 & -1 & 0 & 0 & s_3 \\ 0 & 0 & -1 & 0 & s_2 \\ 0 & 0 & 0 & -1 & s_1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \tag{20}$$

where s_n is the sum of the nth powers of the complex roots. Hence P can be calculated by purely rational operations.

There is a transforming matrix of order $n!$ which interchanges A_1 and A_2 and leaves $A_3,\ldots,A_n$ unaltered, the A's being regarded as matrices of order $n!$

$$A_3 = \begin{bmatrix} q_1 & -q_2 & q_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let

$$S = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

where each of the elements α, β, γ interchanges A_1 and A_2.

* A. R. Richardson, *Proc. London. Math. Soc.* (2), 28 (1929), 416.

Now $SA_3 = A_3 S$ requires that

$$\alpha_1 q_1 + \alpha_2 = q_1\alpha_1 - q_2\beta_1 + q_3\gamma_1, \qquad -\alpha_1 q_2 + \alpha_3 = q_1\alpha_2 - q_2\beta_2 + q_3\gamma_2,$$
$$\beta_1 q_1 + \beta_2 = \alpha_1, \qquad -\beta_1 q_2 + \beta_3 = \alpha_2,$$
$$\gamma_1 q_1 + \gamma_2 = \beta_1, \qquad -\gamma_1 q_2 - \gamma_3 = \beta_2,$$
$$-\alpha_1 q_3 = q_1\alpha_3 - q_2\beta_3 + q_3\gamma_3$$
$$-\beta_1 q_3 = \alpha_3$$
$$-\gamma_1 q_3 = \beta_3.$$

The solution is $\alpha_1 = \beta_2 = \gamma_3$, $\alpha_2 = \alpha_3 = \beta_1 = \beta_3 = \gamma_1 = \gamma_2 = 0$.

Hence $S = \begin{bmatrix} P & & \\ & P & \\ & & P \end{bmatrix}$ is the type of solution.

A repetition of the argument proves the result. Since $\kappa(A_1)$ is a field, we may apply an induction to $f_n(x)/(x-A_1)$ and then rewrite the elements of the transforming matrices of order $(n-1)!$ as scalar matrices of order n and so prove that matrices T_{ij} exist, of order $n!$, which interchange any pair of roots A_i, A_j whilst leaving the others unaltered. It is evident that the transforming matrix operators $T_{ij}^{-1}(\quad)T_{ij}$ when applied to elements of the realm $\kappa(A_1, A_2,..., A_n)$ form a group which is simply isomorphic with the symmetric group. Further, it is easy to show that any T_{ij} is the product of a *fixed* S_{ij} by a number in the realm $\kappa(V_1)$. The unit element of the group consists of all numbers in $\kappa(V_1)$.

When T exists which transforms every V_i into a direct sum of matrix elements of orders in the leading diagonal, $V_1,..., V_s$ may be chosen so as to have as leading elements a set of conjugate solutions of the same partial resolvent. The matrices T_{ij} which interchange $V_1,..., V_s$ themselves are resolved by T into direct sums of matrices of order s,* and the partial matrices which transform the leading elements of $V_1,..., V_s$ evidently form a group of transforms simply isomorphic with the Galois group of the equation. The relation of this group to the symmetric group is indicated by the way in which T expresses the T_{ij}'s as direct sums of elements not all in the leading diagonal.

Such matrices may themselves be chosen to form a group,† and

* A. Voss, p. 93, not all, however, in the leading diagonal.

† H. Hasse, *Trans. American Math. Soc.* (34), 189 and also Max Deuring, *Algebren*, 60–7, and B. L. van der Waerden, *Gruppen von linearen Transformationen*, 71.

this leads to the interesting conclusion that the matrix roots can be put in a canonical form. For a group of matrices can be put into such a form by transformation and the corresponding matrix roots will then be in canonical form.

Thus for the quadratic $x^2-a_1x+a_2=0$ a canonical form for the Galois group is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and the corresponding canonical roots are

$$\begin{pmatrix}\frac{1}{2}a_1 & \frac{1}{2}(a_1^2-4a_2)\\ \frac{1}{2} & \frac{1}{2}a_1\end{pmatrix},\qquad \begin{pmatrix}\frac{1}{2}a_1 & -\frac{1}{2}(a_1^2-4a_2)\\ -\frac{1}{2} & \frac{1}{2}a_1\end{pmatrix}.$$

By using (20) and P we can find a matrix which will transform T_{ij} into the corresponding canonical matrix and therefore the set of conjugate roots into a canonical set. We have proved that

If the Galois group of an irreducible equation is of order s, then a complete set of conjugate rational matrix roots $A_1,\ldots,A_n$, each of order s can be found, by purely rational operations, which are transformed into each other by a group of matrices, each of order s, simply isomorphic with the Galois group. Further, the realm $\kappa(A_1, A_2,\ldots, A_n)$ is simply isomorphic with the Galois field and, if the matrix representation is of order greater than that of the Galois group, and if V_1 is non-derogatory, then $\kappa(V_1)$ may contain divisors of zero.

In Ex. 7 the group of transforming matrices may be taken to be 1 and Q_{12}, Q_{13}, Q_{23}, Q_{123}, Q_{132} respectively as

$$\begin{bmatrix}-1 & -1 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & -1 & -1 & -1 & -1 & 2\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix},\qquad \begin{bmatrix}1 & 2 & 0 & 2 & 3 & -1\\ -1 & -2 & 0 & -1 & -2 & 0\\ 1 & 3 & 1 & 1 & 2 & 0\\ -1 & -3 & -2 & -2 & -5 & -3\\ 1 & 2 & 1 & 1 & 3 & 2\\ -1 & -2 & -1 & -1 & -2 & -1\end{bmatrix},$$

$$\begin{bmatrix}-1 & 1 & 1 & 0 & 1 & 1\\ 1 & 0 & -1 & 1 & 1 & -1\\ -1 & 0 & 2 & -1 & 0 & 3\\ 2 & -1 & -1 & 1 & -1 & -1\\ -1 & 1 & 1 & -1 & 0 & 1\\ 1 & 0 & -1 & 1 & 0 & -2\end{bmatrix},\qquad \begin{bmatrix}1 & 2 & 1 & 0 & 1 & 2\\ -1 & -2 & 0 & -1 & -2 & 0\\ 1 & 2 & 1 & 1 & 3 & 2\\ -2 & -4 & -2 & -1 & -2 & -1\\ 1 & 3 & 1 & 1 & 2 & 0\\ -1 & -2 & -1 & -1 & -2 & -1\end{bmatrix},$$

$$\begin{bmatrix} -1 & -1 & 1 & -2 & -2 & 2 \\ 1 & 0 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 0 & 1 \\ 1 & 0 & -3 & 2 & 0 & -6 \\ -1 & 0 & 2 & -1 & 0 & 3 \\ 1 & 0 & -1 & 1 & 0 & -2 \end{bmatrix}. \tag{21}$$

For the complete set of solutions (9), T transforms the above thus: if

$$Q'_{123} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Q'_{132} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

which together with the unit matrix form a group, then

$$T^{-1}Q_{12}\,T = \begin{bmatrix} 0 & Q'_{123} \\ Q'_{132} & 0 \end{bmatrix}, \qquad T^{-1}Q_{13}\,T = \begin{bmatrix} 0 & Q'_{132} \\ Q'_{123} & 0 \end{bmatrix},$$

$$T^{-1}Q_{23}\,T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad T^{-1}Q_{123}\,T = \begin{bmatrix} Q'_{123}, & 0 \\ 0\ , & Q'_{132} \end{bmatrix},$$

$$T^{-1}Q_{132}\,T = \begin{bmatrix} Q'_{132}, & 0 \\ 0, & Q'_{123} \end{bmatrix}.$$

If we transform (22) into a canonical form, e.g.

$$P_{123} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_{132} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

then a set of solutions in canonical form is

$$A_1 = \frac{1}{14}\begin{bmatrix} -25 & -1 & -9 \\ 4 & -6 & 2 \\ -1 & 5 & 17 \end{bmatrix}, \qquad A_2 = \frac{1}{14}\begin{bmatrix} 17 & -1 & 5 \\ -9 & -25 & -1 \\ 2 & 4 & -6 \end{bmatrix},$$

$$A_3 = \frac{1}{14}\begin{bmatrix} -6 & 2 & 4 \\ 5 & 17 & -1 \\ -1 & -9 & -25 \end{bmatrix},$$

in which the rational discriminant, 7, is put in evidence.

5. Remarks

The characteristic equation of the matrix $g(A_1,\ldots,A_n)$ has the complete set of conjugate roots obtained by operating on the suffixes of the A's by the substitutions of the group of the matrix representation. This offers a simple way of expressing the equation of squared differ-

ences, the resolvent, the general number in the field, the discriminant of the field and similar numbers in determinant form. Further, an ideal can be expressed as a matrix and the theory of classes of ideals and of reciprocity may be regarded as the theory of a set of equations each of which has solutions which are transforming matrices for some of the others. Thus $x^2-m=0$ and $x^2-n=0$ are such equations if m is the norm of a number in $\kappa(\sqrt{n})$ and if n is the norm of a number in $\kappa(\sqrt{m})$, whilst $x^2-m=0$ is self-involutory if the quadratic character $\left(\dfrac{-1}{m}\right)$ is $+1$. The equation $x^3-2=0$ is also self-involutory, for the solutions given in § 2 are transformed into each other by Q_{123}, Q_{132}, i.e.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & -2 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

both of which are solutions of $x^3-2=0$.

Finally, our methods will apply to unilateral equations with matrix coefficients even if non-commutative and also to equations having repeated factors. It is interesting to note too that a solution of $xax+bx+xc+d=0$ is

$$x = \begin{bmatrix} -ba^{-1} & ba^{-1}c-d \\ a^{-1} & -a^{-1}c \end{bmatrix}.$$

PHOTONS, ENERGY, AND RED-SHIFTS IN THE SPECTRA OF NEBULAE

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1. Introduction

IN a recent paper Milne has constructed a complete dynamics of a particle, free or constrained, in the presence of the substratum or 'smoothed-out' universe.* It represented a substantially new attack on the problem of the origin of the laws of dynamics and gravitation: whereas the usual procedure is to extrapolate *laws of nature*, known from observation to hold locally, so as to build up a model universe and so a cosmology, the new procedure is to extrapolate *phenomena*, from which alone laws of nature can be derived, fitting these phenomena together so as to construct a complete model universe and then determining the laws of nature by analysis of the totality of phenomena possible in this model.

The extrapolation of phenomena to form a model universe was originally carried out by means of the 'cosmological principle'.† But it has recently been shown that the elementary properties of systems satisfying the cosmological principle can also be derived by use of the 'sample principle', according to which it is only necessary to ensure that local phenomena are everywhere the same; world-wide pictures are then automatically identical.‡ It is indeed immaterial whether the actual universe obeys the cosmological principle; all we want to know in the long run are the laws of nature holding locally, and we can attack this question by deriving them as the condition that local phenomena are in effect capable of being fitted together to form a self-consistent universe. If the laws of nature so derived did not agree with local experience, then we should have a proof that the whole actual universe did not obey the cosmological principle. The dynamics constructed by this method does, however, agree with

* Milne, *Proc. Roy. Soc.* 154A (1936), 22–52. I wish to acknowledge the considerable assistance which I have received from Professor Milne in the preparation of this note for publication.

† Milne, *Relativity, Gravitation, and World-Structure* (Oxford, 1935), Chap. 3: referred to here as WS.

‡ Whitrow, *Zeits. für Astrophys.* 12 (1936), 47–55. See also WS § 134; Hubble, *Halley Lecture* (1934), 4.

the dynamics observed to hold good locally in the sense that it reduces to the Newton-Einstein dynamics near the observer. But in the new dynamics *energy is an invariant*, not the time-component of a 4-vector, and so is the same for all fundamental observers.

The purpose of the present note is to extend this dynamics to photons, by the usual process of regarding a photon as the limit of a particle of zero rest-mass moving with the velocity of light. The result is to give, without any appeal to the quantum theory of radiation, the fundamental relation* $E = h\nu$. But the astronomical setting of the result is novel. It exhibits the red-shift of the nebulae alternatively either as a Doppler effect or as a gravitational red-shift due to loss of energy as a photon, in moving from a nebula to the observer, runs down the apparent potential-gradient of the substratum. Either interpretation is equally valid, so that the 'reddening due to age', to use the common phrase, can be considered at will as kinematical or dynamical. It will be understood that in speaking here of a gravitational red-shift we are referring to the red-shift due to the field of the substratum or general distribution of nebulae; the general theory of relativity (not here appealed to) would predict a possible additional red-shift, due to the fact that we are near one condensation (our own galactic centre) and that the photon has originated near another, so that there may be an extra (positive or negative) shift due to the consequent additional difference of gravitational potential.

2. The energy of a particle

Let O be an observer associated with a fundamental particle, that is with a particle of the substratum. If O adopts Euclidean space, of coordinates x, y, z with himself as origin, and Newtonian time t, measured from the natural zero of time, then in his description, the substratum has a distribution of particle-density $n\,dxdydz$, where n, the number of particles per unit volume near (x,y,z) at epoch t, according to O, is given by

$$n = \frac{Bt}{c^3(t^2 - \mathbf{P}^2/c^2)^2}, \tag{1}$$

* Kermack, McCrea, and Whittaker, *Proc. Roy. Soc. Edinburgh*, 53 (1932), 31–47, and Synge, *Quart. J. of Math.* (Oxford), 6 (1935), 199–204 have proved that when a photon is emitted by one atom and absorbed by another in a gravitational field the ratio energy/frequency is the same for emission and absorption.

P being the vector (x, y, z), B a constant of zero physical dimensions, and c the speed of light. The velocity **V** of any particle **P** of the system, at epoch t according to O, is given by

$$\mathbf{V} = \mathbf{P}/t. \tag{2}$$

In the presence of this substratum, any free particle moving through position **P** at epoch t with velocity **V** in O's description undergoes an acceleration $d\mathbf{V}/dt$ given by

$$\frac{d\mathbf{V}}{dt} = (\mathbf{P}-\mathbf{V}t)\frac{Y}{X}G(\xi), \tag{3}$$

where $$X = t^2-\mathbf{P}^2/c^2, \quad Y = 1-\mathbf{V}^2/c^2, \quad Z = t-\mathbf{P}\cdot\mathbf{V}/c^2 \tag{4}$$

and $$\xi = \frac{Z^2}{XY} = \frac{(t-\mathbf{P}\cdot\mathbf{V}/c^2)^2}{(t^2-\mathbf{P}^2/c^2)(1-\mathbf{V}^2/c^2)} \geqslant 1. \tag{5}$$

In his new dynamics Milne takes $G(\xi) = -1$ and assigns to *any* particle of rest-mass m moving with an arbitrary velocity **V** through the position **P** at epoch t (all relative to O) the energy $E = mc^2\xi^{\frac{1}{2}}$ and inertial mass $M = m\xi^{\frac{1}{2}}$. For $\mathbf{P} = 0$, i.e. when O is chosen to be at the fundamental particle close to the given particle, E reduces to $mc^2(1-\mathbf{V}^2/c^2)^{-\frac{1}{2}}$, M to $m(1-\mathbf{V}^2/c^2)^{-\frac{1}{2}}$, which are Einstein's expressions. In all cases $E = Mc^2$ as in Einstein's dynamics.

It is readily seen, however, that, if the system admits the concept of energy *as an invariant for all fundamental observers*, then, whatever the value of $G(\xi)$, the energy of a particle must be given by Milne's formula. For the only two independent functions of **P**, **V**, t which are invariants are ξ, X and hence the energy of a particle must be of the form $\phi(\xi, X)$. Consequently the energy of a slow-moving particle in the neighbourhood of O is given by $\phi(\xi, X)$ where $\xi \sim 1$, $X \sim t^2$. It follows that, if our definition of energy is to be compatible with the well-established principle of conservation of energy for slow-moving particles in the neighbourhood of our own galactic centre, then $\phi(\xi, X)$ must be independent of X. Finally, for any particle passing O the energy must be given by Einstein's formula* $mc^2(1-\mathbf{V}^2/c^2)^{-\frac{1}{2}}$. Since the value of ξ for $\mathbf{P} = 0$ is $(1-\mathbf{V}^2/c^2)^{-1}$ we find that $\phi(\xi, X)$ must be of the form $mc^2\xi^{\frac{1}{2}}$.

* Einstein deduced this formula dynamically, whereas the present considerations are purely kinematical. However, if the definition of energy here adopted is to be of physical significance, it is clear that for a particle passing O the expression must be of the same algebraic form as Einstein's.

3. The energy of a photon

We consider a photon as a particle moving with the velocity of light. Milne has shown, from purely kinematical considerations, that such a particle must necessarily contain in its description a parameter, of the dimensions of a frequency, additional to the specification of its position and direction of motion.* Our object is to link this parameter with the energy of the particle, by purely kinematic arguments, i.e. without assuming the quantum theory of radiation.

We first observe that for a particle moving with the speed c the invariant ξ is infinite. Hence, since the equation of motion (3) can be written in the form

$$\frac{d\mathbf{V}}{dt} = (\mathbf{P}-\mathbf{V}t)\frac{Z^2}{X^2}\frac{G(\xi)}{\xi}, \tag{6}$$

a photon may be regarded as a free particle describing a rectilinear trajectory with the velocity of light if and only if

$$\lim_{\xi\to\infty}\frac{G(\xi)}{\xi} = 0. \tag{7}$$

Consider now a photon passing through the fundamental observer O. Let ν be O's measure of its frequency. The photon is to be regarded as the limit of a particle whose speed $|\mathbf{V}| \to c$ and whose rest-mass $m \to 0$. In order that E may have a finite limit, the ratio $m/(1-\mathbf{V}^2/c^2)^{\frac{1}{2}}$ must have a finite limit. This limit must depend on the description of the photon,† i.e. on ν. We put accordingly

$$\mu(\nu) = \lim_{\substack{m\to 0\\ |\mathbf{V}|\to c}}\frac{m}{(1-\mathbf{V}^2/c^2)^{\frac{1}{2}}}. \tag{8}$$

The energy E of the photon, being an invariant, must have the same value for the given photon at its given position and epoch for all fundamental observers. A particular fundamental observer is that one past whom the photon is flying, and we can calculate E by determining its value as calculated by O. Accordingly

$$E = \lim_{\substack{m\to 0\\ \to c}}\left[\frac{mc^2(t-\mathbf{P}\cdot\mathbf{V}/c^2)}{(t^2-\mathbf{P}^2/c^2)^{\frac{1}{2}}(1-\mathbf{V}^2/c^2)^{\frac{1}{2}}}\right]_{\mathbf{P}=0}$$
$$= c^2\mu(\nu). \tag{9}$$

* Milne, *Zeits. für Astrophys.* 6 (1933), 83.

† In effect we are assuming that the energy of a photon is completely specified by its frequency.

Now let a photon be emitted at time t, reckoned by O, from a nebula or fundamental observer O' in the direction of O. Let its energy at O' be E'; let its frequency to O' be ν'; and let its frequency to O be ν. Then by (9)

$$E' = c^2\mu(\nu'). \tag{10}$$

But the same value E' must be assigned to the energy by O, since the energy is an invariant. Putting $OO' = \mathbf{P}_0$ we see that O calculates the energy as

$$\lim_{\substack{m\to 0\\ |\mathbf{V}|\to c}} \left[\frac{mc^2(t-\mathbf{P}_0\cdot\mathbf{V}/c^2)}{(t^2-\mathbf{P}_0^2/c^2)^{\frac{1}{2}}(1-\mathbf{V}^2/c^2)^{\frac{1}{2}}}\right]. \tag{11}$$

In this expression, since the direction of $\mathbf{V}$ is that of the vector $-\mathbf{P}_0$,

$$\mathbf{V} = -c\frac{\mathbf{P}_0}{|\mathbf{P}_0|}. \tag{12}$$

Combining (11) and (12) we have

$$\begin{aligned} E' &= \lim_{\substack{m\to 0\\ |\mathbf{V}|\to c}} \frac{mc^2}{(1-\mathbf{V}^2/c^2)^{\frac{1}{2}}} \frac{t+|\mathbf{P}_0|/c}{(t^2-\mathbf{P}_0^2/c^2)^{\frac{1}{2}}} \\ &= c^2\mu(\nu)\left[\frac{t+|\mathbf{P}_0|/c}{t-|\mathbf{P}_0|/c}\right]^{\frac{1}{2}}. \end{aligned} \tag{13}$$

But $\mathbf{P}_0 = \mathbf{V}_0 t$, where $\mathbf{V}_0$ is the velocity of O'. Hence

$$E' = c^2\mu(\nu)\left[\frac{1+|\mathbf{V}_0|/c}{1-|\mathbf{V}_0|/c}\right]^{\frac{1}{2}}. \tag{14}$$

Equating (10) and (14) we have

$$\mu(\nu') = \mu(\nu)\left[\frac{1+|\mathbf{V}_0|/c}{1-|\mathbf{V}_0|/c}\right]^{\frac{1}{2}}. \tag{15}$$

But by the ordinary kinematic theory of the Doppler effect, established by light-signalling analysis only without recourse to dynamics, the frequency ν observed by O is less than that, ν', observed by the receding O' in the ratio

$$\nu = \nu'\left[\frac{1-|\mathbf{V}_0|/c}{1+|\mathbf{V}_0|/c}\right]^{\frac{1}{2}}. \tag{16}$$

By (15) and (16)

$$\frac{\mu(\nu')}{\nu'} = \frac{\mu(\nu)}{\nu}. \tag{17}$$

Since O is arbitrary the ratio must be the same for all fundamental particles. Call it h/c^2, where h is a universal constant.* Then

$$\mu(\nu) = h\nu/c^2,$$

and so by (9)

$$E = h\nu. \tag{18}$$

* By assuming that for a photon $m/(1-\mathbf{V}^2/c^2) \to \mu(\nu)$ we have necessarily found that h is a secular invariant. The possibility of h varying with the epoch has been discussed by Chalmers and Chalmers, *Phil. Mag.* 19 (1934), 436–46.

We are thus led immediately to the isolation of the quantum constant h.

4. Red-shifts in the spectra of nebulae

The energy of the photon when at O' is $E' = h\nu'$ and when at O its energy is $E = h\nu$. Thus

$$\frac{E'}{E} = \frac{\nu'}{\nu} = \left[\frac{1+|\mathbf{V}_0|/c}{1-|\mathbf{V}_0|/c}\right]^{\frac{1}{2}}. \tag{19}$$

Consequently the energy of the photon *diminishes* as it passes from O' to O. Its frequency, either to O' or to O, remains constant during this passage, but the invariant energy decreases from E' to E. It advances with constant speed and so we may say that its potential energy in the gravitational field of the substratum diminishes as it passes from O' to O, for Milne has shown that in his revised dynamics the invariant expression E counts in both what we are disposed to call kinetic energy and what we are disposed to call gravitational potential energy in the field of the substratum. Thus we have at will either a kinematical or dynamical description of the red-shift.

It must be noticed that in the present formulation the quantum relation $E = h\nu$ is not an invariant relation. E is an invariant, the same for all observers at any given event in the photon's history; ν is different (in general) for every different observer. E is constant for all observers for a given photon at an assigned position; ν is constant for a given photon for an assigned observer. The equality $E = h\nu$ is just satisfied for each observer as the photon passes that observer.

It may be repeated that the history of a photon as here analysed is that of a particle moving with the speed of light in the smoothed-out model. We have thus taken into account only the *general* gravitational field of the universe. The effect of condensations would require a separate treatment.

5. Summary

The revised dynamics recently derived by Milne from purely kinematical considerations is applied to photons. The relation $E = h\nu$ is shown to follow, by considering the passage of a photon from a nebula to the observer in the gravitational field of the smoothed-out universe. It is further shown that the red-shift may be considered either as a kinematic Doppler effect, or as a loss of energy consequent on passage through the gravitational difference of potential between nebula and observer.

THE AUTOMORPHIC TRANSFORMATIONS OF A SINGULAR MATRIX PENCIL ✓

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1. In the theory of canonical matrices two matrix pencils

$$\Gamma_1 = \rho A_1 + \sigma B_1 \text{ and } \Gamma_2 = \rho A_2 + \sigma B_2,$$

each with m rows and n columns, are said to be *equivalent* if two constant non-singular matrices P and Q of degrees m and n exist such that

$$P\Gamma_1 Q = \Gamma_2 \qquad (|P| \neq 0;\ |Q| \neq 0). \tag{1}$$

The fundamental result is that two pencils can be reduced to the same canonical form, if and only if they are equivalent. Suppose now that Γ_1 and Γ_2 are equivalent; then it is possible to solve (1) for P and Q, and the natural question arises: What is the most general solution of this equation? It is easily seen that this problem is equivalent to finding the most general pair of matrices P, Q which transforms a given pencil $\Gamma = \rho A + \sigma B$ into itself, i.e., which satisfies the equation

$$P\Gamma_1 Q = \Gamma_1, \tag{2}$$

or, comparing coefficients of ρ and σ,

$$PAQ = A; \tag{2a}$$

$$PBQ = B. \tag{2b}$$

In this case we shall say that the pair of matrices (P, Q) is an *automorphic* transformation of Γ, and we propose to determine all such automorphic transformations; in particular, we shall express the number of linearly independent ones among them in terms of the invariants of Γ, i.e., in terms of the invariant factors and the Kronecker indices of Γ.

We may make some convenient assumptions regarding

$$\Gamma = \rho A + \sigma B,$$

without restricting the generality of the investigations.

First, we shall assume that A is of the same rank as Γ itself, a condition which can always be fulfilled by a linear transformation of the variables ρ, σ. For, let ρ_0, σ_0 be such that

$$A_1 = \rho_0 A + \sigma_0 B$$

is of maximum rank, i.e., of the same rank as Γ itself, and put

$$B_1 = \rho_1 A + \sigma_1 B,$$

where ρ_1 and σ_1 are only subject to the condition

$$\begin{vmatrix} \rho_0 & \rho_1 \\ \sigma_0 & \sigma_1 \end{vmatrix} \neq 0. \tag{3}$$

If we now introduce new variables ρ', σ' by the transformation

$$\rho = \rho_0 \rho' + \rho_1 \sigma'$$
$$\sigma = \sigma_0 \rho' + \sigma_1 \sigma',$$

we see that Γ can be written as

$$\Gamma = \rho' A_1 + \sigma' B_1,$$

and that it now has the property required. Again, the automorphic transformations remain the same, since the equations

$$PA_1 Q = A_1, \tag{4a}$$
$$PB_1 Q = B_1 \tag{4b}$$

are equivalent to (2a) and (2b) on account of (3). After this preliminary remark we shall replace the two homogeneous variables ρ, σ by *one* variable, λ, and write the pencil in the form

$$\Gamma = \lambda A + B,$$

where A is of the same rank as Γ.

Next, Γ may be replaced by any pencil Γ_0 which is equivalent to Γ. For, let

$$\Gamma_0 = S \Gamma T \quad (|S| \neq 0; \quad |T| \neq 0).$$

If (P, Q) is an automorphic transformation of Γ, then

$$(SPS^{-1}, T^{-1}QT)$$

is an automorphic transformation of Γ_0, and vice versa. Thus a (1, 1) correspondence is established between the automorphic transformations of Γ and Γ_0. In particular, we may assume that Γ_0 is the canonical form* of Γ, which we write in the form

$$\Gamma_0 = \Gamma_1 \dot{+} \Gamma_2 \dot{+} \Gamma_3, \tag{5}$$

where $$\Gamma_1 = \dot{\sum_{i=1}^{p}} (\alpha_i \Lambda_{m_i}) \quad (0 < m_1 < m_2 < \ldots < m_p), \tag{5.1}$$

$$\Gamma_2 = \lambda I_k + M_k, \tag{5.2}$$

$$\Gamma_3 = \dot{\sum_{j=1}^{q}} (\beta_j \Lambda_{n_j}) \quad (0 < n_1 < n_2 < \ldots < n_q), \tag{5.3}$$

using the 'direct sum' of matrices which is defined as follows:

* See the footnote on the next page.

$$\Gamma_1 \dot{+} \Gamma_2 \dot{+} \ldots \dot{+} \Gamma_r = \sum_{i=1}^{r} {}^{\cdot}\, \Gamma_i = \operatorname{diag}(\Gamma_1, \Gamma_2, \ldots, \Gamma_r) = \begin{bmatrix} \Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_r \end{bmatrix};$$

and, when $\Gamma_1 = \Gamma_2 = \ldots = \Gamma_r = \Gamma$, we write

$$\Gamma \dot{+} \Gamma \dot{+} \ldots \dot{+} \Gamma = (r\Gamma).$$

In (5.1) and (5.3), Λ_{m_i} is the typical singular sub-matrix* corresponding to a row vector of minimal degree m_i annihilating Γ, e.g.

$$\Lambda_1 = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} \lambda & . \\ 1 & \lambda \\ . & 1 \end{bmatrix}, \ldots,$$

and in general

$$\Lambda_s = \begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \\ & & \lambda \\ & & 1 \end{bmatrix}_{s+1,s}. \tag{6}$$

We remark that Λ_s has $s+1$ rows and s columns. The pencil Γ_2 in (5.2) is the 'non-singular core of Γ' of type $k \times k$, say, and there is no loss of generality in assuming that the matrix coefficient in Γ_2 is the unit matrix; for since that coefficient is of the same rank as Γ_2, it must be non-singular and may be removed as a matrix factor.

The Kronecker minimal indices are exhibited in the canonical form, viz:

α_1 times m_1, α_2 times m_2,... for the first set,

β_1 times n_1, β_2 times n_2,... for the second set.

These numbers together with the invariant factors of Γ_2 are the invariants of Γ: they are the same for all pencils equivalent to Γ. In what follows we shall assume that Γ is already in canonical form, i.e. $\Gamma_0 = \Gamma$.

Since $|Q| \neq 0$, we may put $R = Q^{-1}$ and write (2) as

$$P\Gamma = \Gamma R \quad (|P| \neq 0;\ |R| \neq 0) \tag{7}$$

or by (5)
$$P(\Gamma_1 \dot{+} \Gamma_2 \dot{+} \Gamma_3) = (\Gamma_1 \dot{+} \Gamma_2 \dot{+} \Gamma_3)R. \tag{8}$$

We now partition P and R in conformity with the three isolated sub-matrices of Γ, i.e. we put

$$P = [P_{ij}], \qquad R = [R_{ij}] \qquad (i,j = 1, 2, 3).$$

* See Turnbull and Aitken, *Canonical Matrices* (Glasgow 1932), 121; and W. Ledermann, *Proc. Edinburgh Math. Soc.* (2) 4 (1934), 92–105.

Hence (8) resolves into nine partial equations

$$P_{ij}\Gamma_j = \Gamma_i R_{ij},$$

or, in full,

$$\left.\begin{array}{lll} \text{(i) } P_{11}\Gamma_1 = \Gamma_1 R_{11}, & \text{(iv) } P_{12}\Gamma_2 = \Gamma_1 R_{12}, & \text{(vii) } P_{13}\Gamma_3 = \Gamma_1 R_{13}, \\ \text{(ii) } P_{21}\Gamma_1 = \Gamma_2 R_{21}, & \text{(v) } P_{22}\Gamma_2 = \Gamma_2 R_{22}, & \text{(viii) } P_{23}\Gamma_3 = \Gamma_2 R_{23}, \\ \text{(iii) } P_{31}\Gamma_1 = \Gamma_3 R_{31}, & \text{(vi) } P_{32}\Gamma_2 = \Gamma_3 R_{32}, & \text{(ix) } P_{33}\Gamma_3 = \Gamma_3 R_{33}, \end{array}\right\} \quad (9)$$

and it is the object of the following pages to give a complete solution to these equations, the m^2+n^2 elements of P and R being regarded as the unknowns. Only equation (v) seems to have received attention in the literature. For substituting for Γ_2 from (5.2) and suppressing unnecessary indices we can write this equation as

$$P(\lambda I+M) = (\lambda I+M)R,$$

whence

$$P = R$$

and

$$PM = MR.$$

Hence

$$PM = MP.$$

The solution of (v), therefore, involves the finding of the most general matrix P that commutes with a given matrix M.

This problem was first solved by Frobenius and has since been treated by several authors.* It was found that the number of linearly independent solutions of (v) is

$$t_5 = t = e_1+3e_2+5e_3+7e_4+\dots, \tag{10}$$

where e_ν is the degree in λ of the νth invariant factor of the matrix λI_k+M_k. In what follows we shall obtain similar results for the remaining eight equations of (9) to which we shall refer later simply by Roman numerals. The total number t of parameters in the general solution of (7) is equal to the sum of the sub-totals $t_1, t_2,\dots, t_9$, giving the numbers of parameters in the solutions of those nine equations (9). It will be found that three of these numbers are zero, i.e., that the corresponding equations have only the trivial solution in which all unknowns vanish; and the final result will be

$$t = \sum_{s=1}^{9} t_s = \tau+\sum_{i\geqslant j}\alpha_i\alpha_j(m_i-m_j+1)+\sum_{j\geqslant i}\beta_i\beta_j(n_j-n_i+1)+$$
$$+k(\sum_i \alpha_i+\sum_j \beta_j)+\sum_{i,j}\alpha_i\beta_j(m_i+m_j).$$

* Frobenius, *Berliner Sitzungsb.* (1910), 3–15, where other references may be found; also D. E. Rutherford, *Proc. Ak. Wet. Amsterdam*, 35 (1932), 870–5.

2. Before solving the commutantal equations § 1 (9), we shall make some remarks on the typical singular sub-matrix (§ 1 (6)):

$$\Lambda_s = \begin{bmatrix} \lambda & & & \\ 1 & \lambda & & \\ & 1 & & \\ & \cdot & & \lambda \\ & & & 1 \end{bmatrix} = \lambda \begin{bmatrix} I_s \\ \cdot \end{bmatrix} + \begin{bmatrix} \cdot \\ I_s \end{bmatrix}, \tag{1}$$

where I_s is the unit matrix of degree s, and the dot below or above it indicates a row of zeros.

First, we wish to determine the row vectors and column vectors that annihilate Λ_s. Let

$$x = \{\xi_1, \xi_2, \ldots, \xi_s\}$$

be a column vector with s elements and

$$y = [\eta_0, \eta_1, \ldots, \eta_s]$$

a row vector with $s+1$ elements. We then prove the following lemma:

Lemma 1. *The equation*

$$\Lambda_s x = 0 \tag{2}$$

admits only of the solution $x = 0$, i.e. the columns of Λ_s are linearly independent; the general solution of

$$y\Lambda_s = 0 \tag{3}$$

is

$$y = \phi(\lambda)[1, -\lambda, (-\lambda)^2, \ldots, (-\lambda)^s],$$

where $\phi(\lambda)$ is an arbitrary function of λ and

$$u_s \equiv [1, -\lambda, (-\lambda)^2, \ldots, (-\lambda)^s]$$

is a vector of degree s.

Proof. The rank of Λ_s is s, because, on cancelling the first row of Λ_s, we obtain a minor of order s identically equal to unity. In the set of homogeneous equations (2), the number of unknowns therefore equals the rank, and by the fundamental theorem of linear equations the system has only the trivial solution $x = 0$. In the set of equations (3), however, the number of unknowns exceeds the rank by unity and the most general solution is a scalar multiple of any particular solution which may be taken to be the vector u_s defined above, since

$$u_s\Lambda_s = 0, \tag{4}$$

as is easily verified.

LEMMA 2. *If P is a constant matrix with $s+1$ rows, then the equation*

$$u_s P = 0 \tag{5}$$

is impossible unless

$$P = 0.$$

The proof follows immediately on writing out (5) in full and equating the coefficients of $1, \lambda, \lambda^2,..., \lambda^s$ to zero.

3. We now turn to the discussion of equation § 1 (9). Consider the simple case

$$P\Lambda_s = \Lambda_{s'} R, \tag{1}$$

which is solved by the following theorem.

THEOREM 1. *When $s' < s$, equation* (1) *has only the trivial solution*

$$P = 0, \qquad R = 0; \tag{2}$$

when $s' \geqslant s$ and $s'-s = d$ (say), then the general solution is

$$P = [\phi_{i-j}]_{s'+1,s+1} \quad \begin{pmatrix} i = 0, 1,..., s' \\ j = 0, 1,..., s \end{pmatrix}; \tag{3}$$

and

$$R = [\phi_{i-j}]_{s',s} \quad \begin{pmatrix} i = 1, 2,..., s' \\ j = 1, 2,..., s \end{pmatrix}; \tag{4}$$

where $\phi_0, \phi_1,..., \phi_d$ are $d+1$ arbitrary constants and

$$\phi_{i-j} = 0 \quad \begin{cases} (i-j > d) \\ (i-j < 0). \end{cases}$$

For instance in the case $s' = 5$, $s = 3$, $d = 2$, we have

$$P = \begin{bmatrix} \phi_0 & \cdot & \cdot & \cdot \\ \phi_1 & \phi_0 & \cdot & \cdot \\ \phi_2 & \phi_1 & \phi_0 & \cdot \\ \cdot & \phi_2 & \phi_1 & \phi_0 \\ \cdot & \cdot & \phi_2 & \phi_1 \\ \cdot & \cdot & \cdot & \phi_2 \end{bmatrix}, \qquad R = \begin{bmatrix} \phi_0 & \cdot & \cdot \\ \phi_1 & \phi_0 & \cdot \\ \phi_2 & \phi_1 & \phi_0 \\ \cdot & \phi_2 & \phi_1 \\ \cdot & \cdot & \phi_2 \end{bmatrix}.$$

Proof. On premultiplying (1) by u_s we obtain by § 2 (4)

$$(u_{s'} P)\Lambda_s = 0,$$

i.e. the vector $u_{s'} P$ annihilates Λ_s. Hence by Lemma 1 (§ 2) we have

$$u_{s'} P = \phi(\lambda)[1, -\lambda, (-\lambda)^2,..., (-\lambda)^s]. \tag{5}$$

The elements of the vector on the left-hand side are polynomials of degree not higher than s'. Hence, comparing the first elements of either side, we see that $\phi(\lambda)$ is also such a polynomial.

Now, when s' is less than s, equation (5) is obviously impossible unless $\phi(\lambda)$ is equal to zero and hence

$$u_{s'} P = 0, \tag{6}$$

which, by Lemma 2 (§ 2), implies

$$P = 0.$$

Equation (1) then becomes $0 = \Lambda_s R$ which entails

$$R = 0,$$

since the columns of Λ_s are linearly independent. This proves the first part of the theorem.

Next, when $s' \geqslant s$, (5) can evidently be solved and $\phi(\lambda)$ will, in general, be a polynomial of degree $d = s'-s$;

$$\phi(\lambda) = \phi_0 - \phi_1\lambda + \phi_2\lambda^2 + ... + \phi_d(-\lambda)^d,$$

say. Putting

$$P = [p_{ij}] \quad (i = 0, 1,..., s'; j = 0, 1,..., s)$$

and comparing the jth elements of either side of (5), we obtain

$$p_{0j}+p_{1j}(-\lambda)+p_{2j}(-\lambda)^2+...+p_{s'j}(-\lambda)^{s'} = \phi_0(-\lambda)^j+\phi_1(-\lambda)^{j+1}+$$
$$+\phi_2(-\lambda)^{j+2}+...+\phi_d(-\lambda)^{j+d} \quad (j = 0, 1, 2,..., s).$$

Hence

$$p_{0j} = p_{1j} = ... = p_{j-1,j} = 0,$$
$$p_{jj} = \phi_0, \quad p_{j+1,j} = \phi_1, \quad ..., \quad p_{j+d,j} = \phi_d,$$
$$p_{j+d+1,j} = ... = p_{s'j} = 0;$$

or

$$p_{ij} = \begin{cases} 0 & (i-j < 0), \\ \phi_{i-j} & (0 \leqslant i-j \leqslant d), \\ 0 & (i-j > d), \end{cases}$$

which proves the statement (3) regarding P. In order to determine R we substitute § 2 (1) in (1):

$$P\left\{\lambda\begin{bmatrix} I_{s'} \\ . \end{bmatrix} + \begin{bmatrix} . \\ I_{s'} \end{bmatrix}\right\} = \left\{\lambda\begin{bmatrix} I_s \\ . \end{bmatrix} + \begin{bmatrix} . \\ I_s \end{bmatrix}\right\} R.$$

Comparing the constant terms we get

$$P\begin{bmatrix} . \\ I_{s'} \end{bmatrix} = \begin{bmatrix} . \\ I_s \end{bmatrix} R,$$

which, on premultiplication by $[.\ I_s]$, becomes

$$[.\ I_s] P \begin{bmatrix} . \\ I_s \end{bmatrix} = R.$$

The matrix R is thus expressed in terms of P, and it is readily seen that R is obtained by cancelling the first row and column of P. This proves equation (4). It is easily verified that (4) and (5) are also sufficient conditions that P and R should satisfy (1).

Corollary. When $s' = s$, the general solution of

$$P\Lambda_s = \Lambda_s R$$

is

$$P = \phi_0 I_{s+1}, \qquad R = \phi_0 I_s,$$

involving one parameter ϕ_0. The solution is non-singular, if and only if $\phi_0 \neq 0$.

We now come to the equation

$$P(\alpha\Lambda_s) = (\alpha'\Lambda_{s'})R, \tag{7}$$

where $(\alpha\Lambda_s) = \Lambda_s \dot{+} \Lambda_s \dot{+} \ldots \dot{+} \Lambda_s$ (α times repeated). Writing P and R as

$$P = [P_{ij}], \qquad R = [R_{ij}] \qquad \begin{pmatrix} i = 1, 2,\ldots, \alpha' \\ j = 1, 2,\ldots, \alpha \end{pmatrix},$$

we see that (7) resolves into $\alpha\alpha'$ matrix equations

$$P_{ij}\Lambda_s = \Lambda_{s'} R_{ij} \tag{8}$$

of the type which we have just considered. Hence, if $s' < s$, (8), and therefore (7), is impossible, and, if $s' \geqslant s$, each equation (8) has $s'-s+1$ linearly independent solutions, and the number of parameters in the general solution (7) is consequently

$$\alpha\alpha'(s'-s+1).$$

In particular, when $s' = s$ and $\alpha' = \alpha$, P_{ij} and R_{ij} must be of the form

$$P_{ij} = \phi_{ij} I_{s+1} \qquad R_{ij} = \phi_{ij} I_s \qquad (i,j = 1, 2,\ldots, \alpha),$$

where the ϕ_{ij} are α^2 constants (Corollary to Theorem 1). Introducing a matrix

$$\Phi = [\phi_{ij}],$$

we can write the result in this case as

$$P = [\phi_{ij} I_{s+1}] = \Phi \times I_{s+1},$$
$$R = [\phi_{ij} I_s] = \Phi \times I_s,$$

using the familiar notation for Zehfuss (or Kronecker) matrices. The solutions P and Q are non-singular, if and only if Φ is non-singular.

This enables us at last to obtain the complete solution of (i) (§ 1 (9)):

THEOREM 2. *The general solution of*

$$P\Big[\sum_{i=1}^{p}\cdot(\alpha_i\Lambda_{m_i})\Big]=\Big[\sum_{i=1}^{p}\cdot(\alpha_i\Lambda_{m_i})\Big]R \tag{9}$$

is of the form

$$P=\begin{bmatrix} P_{11} & & & \\ P_{21} & P_{22} & & \\ . & . & . & \\ P_{p1} & P_{p2} & \dots & P_{pp}\end{bmatrix},\qquad R=\begin{bmatrix} R_{11} & & & \\ R_{21} & R_{22} & & \\ . & . & . & \\ R_{p1} & R_{p2} & \dots & R_{pp}\end{bmatrix},$$

involving in all

$$t_1=\sum_{i\geqslant j}\alpha_i\,\alpha_j(m_i-m_j+1) \tag{10}$$

arbitrary parameters.

The proof is obvious from the preceding results. For, again, we partition P and R in a suitable way so as to resolve (9) into the equations

$$P_{ij}(\alpha_j\Lambda_{mj})=(\alpha_i\Lambda_{m_i})R_{ij},$$

which have already been discussed. When $i<j$, we have § 1 (5.1) and therefore $P_{ij}=0$; $R_{ij}=0$. Further, P_{ii} and R_{ii} are of the form $P_{ii}=\Phi_i\times I_{m_i+1}$; $R_{ii}=\Phi_i\times I_m$; and P and R are non-singular if and only if $\Phi_1, \Phi_2,\dots,\Phi_p$ are non-singular.

The solution of (ix) (§ 1 (9)), namely

$$P\Big[\sum_{j=1}^{q}\cdot(\beta_j\Lambda'_{n_j})\Big]=\Big[\sum_{j=1}^{q}\cdot(\beta_j\Lambda'_{n_j})\Big]R,$$

does not involve any new difficulties, since the transposition of this equation leads us back to the previous case. The number of parameters in the general solution of (ix) is therefore

$$t_9=\sum_{j\geqslant i}\beta_i\,\beta_j(n_j-n_i+1). \tag{11}$$

We shall now consider (iv), viz. $P\Gamma_2=\Gamma_1 R$,

i.e.
$$P\Gamma_2=\Big[\sum_i\cdot(\alpha_i\Lambda_{m_i})\Big]R \tag{12}$$

where
$$\Gamma_2=\lambda I_k+M_k$$

is a non-singular pencil of type $k\times k$. As before, (12) resolves into a number of equations of the kind

$$P_i\,\Gamma_2=\Lambda_{s_i}R_i\quad\Big(i=1,2,\dots,\sum_j\alpha_j\Big) \tag{13}$$

where s_i stands for m_1 (α_1 times repeated), m_2 (α_2 times repeated), etc. It is easy to see that (13) is impossible, because premultiplying it by u_{s_i} we get

$$(u_{s_i}P_i)\Gamma_2=0.$$

Since Γ_2 is a non-singular pencil, it follows that $u_{s_i} P_i = 0$, and hence, by Lemma 2 (§ 2), $P_i = 0$. Equation (13) now becomes

$$0 = \Lambda_{s_i} R_i,$$

which entails

$$R_i = 0,$$

the columns of Λ_{s_i} being independent. The solution of (iv) therefore contributes no parameters, i.e.

$$t_4 = 0. \tag{14}$$

We get a different result, however, when Γ_1 and Γ_2 are interchanged, as is the case in (ii), viz.

$$P\Gamma_1 = \Gamma_2 R,$$

i.e.

$$P\Big[\sum_i \cdot (\alpha_i \Lambda_{m_i})\Big] = (\lambda I_k + M_k)R, \tag{15}$$

which may be resolved into $\sum \alpha_j$ equations of this kind:

$$P_i \Lambda_{s_i} = P_i\left\{\lambda\begin{bmatrix} I_{s_i} \\ \cdot \end{bmatrix} + \begin{bmatrix} \cdot \\ I_{s_i} \end{bmatrix}\right\} = (\lambda I_k + M_k)R_i \quad \Big(i = 1, 2, ..., \sum_j \alpha_j\Big), \tag{15'}$$

where s_i takes the values m_1, m_2,..., m_p, the m_i being repeated α_i times. Hence comparing coefficients of λ we get

$$\left.\begin{aligned} P\begin{bmatrix} I_{s_i} \\ \cdot \end{bmatrix} &= R, \\ P\begin{bmatrix} \cdot \\ I_{s_i} \end{bmatrix} &= M_k R, \end{aligned}\right\} \tag{16}$$

where, for simplicity, we have dropped the suffixes of the matrices P and R. Let

$$P = [p_0, p_1, ..., p_{s_i}],$$
$$R = [r_1, r_2, ..., r_{s_i}]$$

introducing column vectors p_0, p_1,..., r_1, r_2,... for the columns of P and R each having k rows. It is easy to see that postmultiplication by $\begin{bmatrix} I_{s_i} \\ \cdot \end{bmatrix}$ or $\begin{bmatrix} \cdot \\ I_{s_i} \end{bmatrix}$ has the effect of cancelling the last or the first row. Hence equation (16) becomes

$$[p_0, p_1, ..., p_{s_i-1}] = [r_1, r_2, ..., r_{s_i}],$$
$$[p_1, p_2, ..., p_{s_i}] = [M_k r_1, M_k r_2, ..., M_k r_{s_i}],$$

and by eliminating r_1, r_2,..., r_{s_i}

$$[p_1, p_2, ..., p_{s_i}] = [M_k p_0, M_k p_1, ..., M_k p_{s_i-1}],$$

whence

$$p_1 = M_k p_0,\quad p_2 = M_k p_1 = M_k^2 p_0,\ldots,\quad p_\nu = M_k^\nu p_0,\ldots$$
$$(\nu = 0, 1,\ldots, s_i)$$

$$r_1 = p_0,\quad r_2 = M_k p_0,\ldots,\quad r_\mu = M_k^{\mu-1} p_0,\ldots\quad (\mu = 1, 2,\ldots, s_i),$$

where the k elements of the vector p_0 remain arbitrary. We have therefore:

THEOREM 3. *The general solution of the equation*

$$P_i \Lambda_{s_i} = (\lambda I_k + M_k) R_i$$

is of the form

$$P_i = [p_0, M_k p_0, M_k^2 p_0,\ldots, M_k^{s_i} p_0],$$
$$R_i = [p_0, M_k p_0, M_k^2 p_0,\ldots, M_k^{s_i-1} p_0],$$

involving k arbitrary parameters, namely the elements of p_0.

Since (15), or (ii), is equivalent to $\sum \alpha_j$ equations of type (15′), we have at once

$$t_2 = k \sum \alpha_i. \tag{17}$$

As before, we see that (vi) (§ 1 (9)), i.e.

$$P\Gamma_2 = \Gamma_3 R$$

and (viii), i.e.

$$P\Gamma_3 = \Gamma_2 R$$

are merely different forms of (ii) and (iv) to which they are reduced by transposition. Γ_3 is then replaced by a pencil of the same type as Γ_1 (§ 1 (5.1) and (5.3)) and the order of the factors is reversed, while Γ_2 and Γ_2' play the same role since the only property we have used was that Γ_2 was a non-singular pencil of type $k \times k$.

By analogy, we obtain therefore:

$$t_6 = k \sum_{j=1}^{q} \beta_j, \tag{18}$$

$$t_8 = 0. \tag{19}$$

We shall now show that (vii), i.e.

$$P\Gamma_3 = \Gamma_1 R,$$

i.e.

$$P\Big[\sum_j{}^{\cdot} (\beta_j \Lambda'_{n_j})\Big] = \Big[\sum_i{}^{\cdot} (\alpha_i \Lambda_{m_i})\Big] R \tag{20}$$

is satisfied only in the trivial case $P = 0$, $R = 0$. For (20) splits up into a number of equations

$$P_{ij} \Lambda'_{n_j} = \Lambda_{m_i} R_{ij} \tag{21}$$

which after premultiplication by u_{m_i} yield

$$(u_{m_i} P_{ij}) \Lambda'_{n_j} = 0,$$

whence

$$u_{m_i} P_{ij} = 0,$$

the *rows* of Λ'_{n_j} being linearly independent (§ 2, Lemma 2). Hence, as before, it follows that

$$P_{ij} = 0 \quad \text{and} \quad R_{ij} = 0.$$

We therefore have
$$t_7 = 0. \tag{22}$$

It remains to solve (iii), which we write as

$$P\Big[\sum_i \cdot (\alpha_i \Lambda_{m_i})\Big] = \Big[\sum_j \cdot (\beta_j \Lambda_{n_j})\Big] R \tag{23}$$

and which reduces to $\Big(\sum_i \alpha_i\Big)\Big(\sum_j \beta_j\Big)$ partial equations

$$P_{ij} \Lambda_{m_i} = \Lambda'_{n_j} R_{ij}, \tag{24}$$

each of them occurring $\alpha_i \beta_j$ times. Substituting for Λ_{m_i} and Λ'_{n_j} from § 2 (1), we get

$$P_{ij}\left\{\lambda\begin{bmatrix} I_{m_i} \\ \cdot \end{bmatrix} + \begin{bmatrix} \cdot \\ I_{m_i} \end{bmatrix}\right\} = \{\lambda[I_{n_j} \cdot] + [\cdot I_{n_j}]\} R_{ij},$$

i.e.

$$P_{ij}\begin{bmatrix} I_{m_i} \\ \cdot \end{bmatrix} = [I_{n_j} \cdot] R_{ij}, \tag{25}$$

$$P_{ij}\begin{bmatrix} \cdot \\ I_{m_i} \end{bmatrix} = [\cdot I_{n_j}] R_{ij}. \tag{26}$$

Let

$$P_{ij} = [p_{\mu\nu}] \qquad \begin{pmatrix} \mu = 1, 2, \ldots, n_j \\ \nu = 1, 2, \ldots, m_{i+1} \end{pmatrix},$$

$$R_{ij} = [r_{\rho\sigma}] \qquad \begin{pmatrix} \rho = 1, 2, \ldots, n_{j+1} \\ \sigma = 1, 2, \ldots, m_i \end{pmatrix}.$$

Then (25) and (26) become

$$p_{\mu\nu} = r_{\mu\nu} \qquad \begin{pmatrix} \mu = 1, 2, \ldots, n_j \\ \nu = 1, 2, \ldots, m_i \end{pmatrix}, \tag{27}$$

$$p_{\mu,\nu+1} = r_{\mu+1,\nu} \qquad \begin{pmatrix} \mu = 1, 2, \ldots, n_j \\ \nu = 1, 2, \ldots, m_i \end{pmatrix}. \tag{28}$$

From these two equations we infer

$$p_{\mu,\nu+1} = p_{\mu+1,\nu}$$

or, replacing ν by $\nu-1$ and iterating the equation,

$$p_{\mu\nu} = p_{\mu+1,\nu-1} = \ldots = p_{\mu+h,\nu-h} = \ldots = \phi_{\mu+\nu-2} \quad \text{say,}$$

i.e. the value of $p_{\mu\nu}$ depends only on the sum of the suffixes, and similarly

$$r_{\mu\nu} = r_{\mu+1,\nu-1} = \ldots = r_{\mu+h,\nu-h} = \ldots = p_{\mu\nu} = \phi_{\mu+\nu-2}.$$

We have therefore proved the following theorem:

THEOREM 4. *The general solution of* $P_{ij}\Lambda_{m_i} = \Lambda'_{n_j} R_{ij}$ *is of the form*

$$P_{ij} = \begin{bmatrix} \phi_0 & \phi_1 & . & . & . & \phi_{m_i} \\ \phi_1 & \phi_2 & . & . & . & \phi_{m_i+1} \\ . & . & . & . & . & . \\ \phi_{n_j-1} & \phi_{n_j} & . & . & . & \phi_{m_i+n_j-1} \end{bmatrix}_{n_j, m_i+1},$$

$$R_{ij} = \begin{bmatrix} \phi_0 & \phi_1 & . & . & . & \phi_{m_i-1} \\ \phi_1 & \phi_2 & . & . & . & \phi_{m_i} \\ . & . & . & . & . & . \\ \phi_{n_j} & \phi_{n_j+1} & . & . & . & \phi_{m_i+n_j-1} \end{bmatrix}_{n_j+1, m_i},$$

involving m_i+n_j *parameters* $\phi_0, \phi_1,..., \phi_{m_i+n_j-1}$.

The number of parameters that occur in the general solution of (iii) is therefore

$$t_3 = \sum_{i,j} \alpha_i \beta_j (m_i+n_j) \qquad \begin{pmatrix} i = 1, 2,..., p \\ j = 1, 2,..., q \end{pmatrix}. \tag{29}$$

We now add up the sub-totals $t_1, t_2,..., t_9$ as given in § 1 (10) and § 3 (10), (11), (14), (17), (18), (19), (22), and (29), and we find that the number of linearly independent solutions of

$$P\Gamma = \Gamma R \tag{30}$$

is equal to

$$\begin{aligned} t &= t_1+t_2+...+t_8+t_9 \\ &= t+k\sum \alpha_i+k\sum \beta_j+\sum_{i\geqslant j} \alpha_i\alpha_j(m_i-m_j+1)+ \\ &\qquad +\sum_{i\geqslant j} \beta_i\beta_j(n_j-n_i+1)+\sum_{i,j} \alpha_i\beta_j(m_i+n_j). \end{aligned}$$

Moreover, a method has been given for actually obtaining all matrices P and R satisfying (30). Since

$$t_4 = t_7 = t_8 = 0,$$

it follows that in the scheme § 1 (9)

$$P_{12} = 0,\ R_{12} = 0;\quad P_{13} = 0,\ R_{13} = 0;\quad P_{23} = 0;\quad R_{23} = 0,$$

so that the general solution is of the form

$$P = \begin{bmatrix} P_{11} & . & . \\ P_{21} & P_{22} & . \\ P_{31} & P_{32} & P_{33} \end{bmatrix}, \qquad R = \begin{bmatrix} R_{11} & . & . \\ R_{21} & R_{22} & . \\ R_{31} & R_{32} & R_{33} \end{bmatrix}.$$

Further, P and R are non-singular if and only if the matrices P_{11}, P_{22}, P_{33} and R_{11}, R_{22}, R_{33} are non-singular, and we have already found the necessary and sufficient conditions that those matrices should be non-singular (see p. 285).

PATH-GEOMETRY AND COSMOGONY

By D. D. KOSAMBI (*Poona*)

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THE equations proposed by E. A. Milne (**1, 2**) for the movement of free particles in his expanding universe may be written in the form

$$\frac{d^2x^i}{d(x^0)^2}-\left(x^i-x^0\frac{dx^i}{dx^0}\right)\frac{1-\sum\left(\frac{dx^i}{dx^0}\right)^2}{(x^0)^2-\sum(x^i)^2}G(\xi)=0, \tag{1}$$

$$\xi=\frac{\left(x^0-\sum x^i\frac{dx^i}{dx^0}\right)^2}{\{(x^0)^2-\sum(x^i)^2\}\left\{1-\sum\left(\frac{dx^i}{dx^0}\right)^2\right\}}\quad (i=1,2,3).$$

These are obtained by putting $x^0=ct$ in Milne's original equations. The equations given above may be taken to give the paths of a three-dimensional space and discussed as such by methods outlined elsewhere (**3, 4, 5, 6**), but I shall replace them by an equivalent four-dimensional system in which x^0 appears as an additional coordinate. We change to a new parameter τ by means of the identities

$$\frac{d}{c\,dt}=\frac{d}{dx^0}=\frac{1}{\dot x^0}\frac{d}{d\tau},\quad \frac{1}{c^2}\frac{d^2}{dt^2}=\frac{d^2}{d(x^0)^2}=\frac{1}{(\dot x^0)^2}\frac{d^2}{d\tau^2}-\frac{\ddot x^0}{(\dot x^0)^3}\frac{d}{d\tau}.$$

In these formulae, as also in all those which succeed, the dots over a letter indicate differentiation with respect to the parameter τ. The equations (1) now become

$$\frac{\ddot x^i}{(\dot x^0)^2}-\frac{\ddot x^0}{(\dot x^0)^3}\dot x^i-\left(x^i-\frac{x^0\dot x^i}{\dot x^0}\right)\frac{1-\sum(\dot x^i/\dot x^0)^2}{(x^0)^2-\sum(x^i)^2}G(\xi)=0\quad (i=1,2,3). \tag{2}$$

These are multiplied by $(\dot x^0)^2$, and may then be written as

$$\ddot x^i-\frac{\ddot x^0}{\dot x^0}\dot x^i-\left(x^i-\frac{x^0\dot x^i}{\dot x^0}\right)\frac{\mathfrak{Y}}{\mathfrak{X}}G(\xi)=0\quad (i=1,2,3), \tag{3}$$

where $\qquad \mathfrak{Y}=(\dot x^0)^2-\sum(\dot x^i)^2,\qquad \mathfrak{X}=(x^0)^2-\sum(x^i)^2,$

$\mathfrak{Y}$ and $\mathfrak{X}$ being equivalent respectively to c^2Y and c^2X in Milne's notation. To simplify further, let the tensor g_{ij} have the components

$$g_{00}=1,\qquad g_{ii}=-1,\qquad g_{0i}=g_{i0}=0,\qquad g_{ij}=0\quad (i\neq j). \tag{4}$$

Take p^i to be a four-component contravariant vector which, in the

particular system of coordinates chosen, has the value x^i. Then,

$$p_i = g_{ir}p^r, \qquad \dot{x}_i = g_{ir}\dot{x}^r, \qquad \mathfrak{Z} = p^r\dot{x}_r = \dot{x}^r p_r = g_{ij}\dot{x}^i p^j, \tag{5}$$
$$\mathfrak{X} = p^r p_r = g_{ij}p^i p^j, \qquad \mathfrak{Y} = \dot{x}^r\dot{x}_r = g_{ij}\dot{x}^i\dot{x}^j, \qquad \xi = \frac{\mathfrak{Z}^2}{\mathfrak{X}\mathfrak{Y}}.$$

The indices now and henceforth range over the values 0, 1, 2, 3, and the tensor-summation convention is to be followed throughout for repeated indices; but the use of the tensor g_{ij} for the purpose of raising and lowering indices is to be confined to the two vectors p^i and $\dot{x}^i$ only.

We have only three equations in (3); the fourth, for $\ddot{x}^0$, remains to be specified so as to give the simplest possible system. This is obviously effected by choosing

$$\frac{\dot{x}^i}{\dot{x}^0}\left\{\ddot{x}^0 - \frac{x^0\mathfrak{Y}}{\mathfrak{X}}G(\xi)\right\} = 0, \qquad x^0 = p^0. \tag{6}$$

Setting apart the case $\dot{x}^0 = 0$, which corresponds to an instantaneous change in the three-dimensional universe, we can now combine all four equations as follows:

$$\left.\begin{aligned} \ddot{x}^i + \alpha^i &= 0, \\ \alpha^i &= -\frac{p^i\mathfrak{Y}}{\mathfrak{X}}G(\xi) \end{aligned}\right\} \quad (i = 0, 1, 2, 3). \tag{I}$$

Using the notation and formulae given in my former papers, the differential invariants of the path-space defined by (I) are given *for the kinematic case* $G(\xi) = -1$ by

$$\epsilon^i = \frac{\partial\alpha^i}{\partial t} = \alpha^i_{;j;k;l} = 0,$$
$$\mathfrak{X}^2 P^i_j = -\mathfrak{X}\mathfrak{Y}\delta^i_j + \mathfrak{Y}p^i p_j + \mathfrak{X}\dot{x}^i\dot{x}_j - \mathfrak{Z}p^i\dot{x}_j,$$
$$\mathfrak{X}^2 R^i_{jk} = \mathfrak{X}(\delta^i_k\dot{x}_j - \delta^i_j\dot{x}_k) + p^i(p_j\dot{x}_k - p_k\dot{x}_j),$$
$$\mathfrak{X}^2 R^i_{jk;l} = \mathfrak{X}(\delta^i_k g_{jl} - \delta^i_j g_{kl}) + p^i(p_j g_{kl} - p_k g_{jl}). \tag{7}$$

They may, of course, be calculated for all G. The question of the existence of a metric is of particular interest and has, to a certain extent, been settled by Walker. But his metric $W(\xi, \mathfrak{X})\sqrt{\mathfrak{Y}}$ cannot be valid even if the integral for his function θ (**8**,484) be given a meaning by some sort of a limiting process. I proceed to show that *for no metric can the equations* (I) *represent geodesics when* $G(\xi) = -1$.

In fact, if such a metric exists, it must be homogeneous in $\dot{x}$ (**6**) and equivalent to a Finsler metric or to one homogeneous of degree zero. But such a metric f is to be obtained as a solution of the system of partial differential equations

$$-\alpha^r f_{;r} + \dot{x}^r f_{,r} = 0, \qquad f_{,i} - \tfrac{1}{2}\alpha^r_{;i} f_{;r} = 0. \tag{8}$$

These have the conditions of integrability

$$P^r_i f_{;r} = 0, \qquad R^r_{ij} f_{;r} = 0 \ldots. \tag{9}$$

From the homogeneity of f, we have $\dot{x}^r f_{;r} = kf$. Substituting this in the identity $p^j P^i_j f_{;i} = 0$, it follows that

$$kf = \frac{3}{\mathfrak{X}} p^r f_{;r}. \tag{10}$$

But the restriction $p^j R^r_{ij} f_{;r} = 0$ gives us

$$p^r f_{;r} p_j = \mathfrak{X} f_{;j}. \tag{11}$$

These may be combined to give $f_{;j} = kfp_j/3$, which has the unique solution $f = A(3)^k$. This, however, is a trivial solution, since the condition for regularity $|f_{;i;j}| \neq 0$ is not fulfilled, i.e., the equations (I) cannot represent the geodesics for such a metric.

The general case $G(\xi) \neq -1$ may also be discussed by similar methods, though this will be of no particular value. It may be pointed out, however, that f being any homogeneous metric reducible to one of Finsler's, or of null degree, any arbitrary $\phi(f)$ is also a possible metric. In addition, there exists along the paths (or, for the Finsler space, must be assumed) the integral $f =$ constant. If any metric homogeneous of degree zero exists, it is not included in those deduced by Walker. The space $G(\xi) = -1$ is purely affine, having no metric.

The interest of the equations (I) lies primarily in the fact that they lend themselves to generalization, and are tensor-invariant. As a matter of fact, equations derived from any 'least action' principle have the advantage of being tensor-invariant, this being from a formal point of view the most important use of the calculus of variations. Combination of (I) with the De Donder-Einstein-Mayer equations may be brought about by adding the terms $\Gamma^i_{jk}\dot{x}^j\dot{x}^k$, the Γ^i_{jk} being Christoffel symbols associated with a general tensor g_{ij} which is to replace that in (4). For the Maxwell tensor, the terms $-F^i_r\dot{x}^r$ must also be added to α^i. The vector p^i may be replaced by one of a more general type. In all this there must be the proviso

that a system of coordinates exists—if not simultaneously for the whole space, at least at every point—such that the tensor g_{ij} in them may be expressed in the form (4), the vector p^i having at the same time the components x^i.

It is not clear, however, whether the original field equations given by De Donder, Einstein, and Mayer for g_{ij}, F^i_j, and the curvature tensor formed from the Γ^i_{jk} should be modified, or whether their equivalent may be derived from equations in terms of the new tensors calculated for the modified α^i. Even when a metric does not exist, it might still be possible to construct a purely affine field theory (7). In all this, it must be kept in mind that there are two distinct spaces associated with the equations (I): the space of which (I) represents the paths, and the space of which g_{ij} is the fundamental tensor.

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ON THE LEBESGUE CLASS OF INTEGRAL FUNCTIONS ALONG STRAIGHT LINES ISSUING FROM THE ORIGIN

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1. Introduction. In an interesting paper* on the mean values of functions regular in a strip, Hardy, Ingham, and Pólya have proved that, if the integral

$$J(x) = \int_{-\infty}^{\infty} |f(x+iy)|^p \, dy$$

converges for $x = \alpha$ and $x = \beta$ while $f(x+iy)$, regular in the strip $\alpha \leqslant x \leqslant \beta$, satisfies the condition

$$f(x+iy) = O\{\exp(e^{k|y|})\} \quad \left(0 < k < \frac{\pi}{\beta-\alpha}\right) \tag{1}$$

uniformly in the strip, then $J(x)$ converges for each x in $\alpha \leqslant x \leqslant \beta$ and $J(x) \leqslant \max[J(\alpha), J(\beta)]$; it is also shown that $\log J(x)$ is a convex function of x.

1.1. In this paper similar results are proved for an angle† under a less restrictive condition (C) (§ 1.3) than (1). In § 2 the results are proved for restricted values of p by using the properties of analytic functions alone. In § 3 these results are extended to any $p > 0$ by using the theory of sub-harmonic functions. The methods of this paper can be applied to prove the convexity property as well, though this point is not discussed in the body of the paper.‡

* Hardy, Ingham, and Pólya, *Proc. Royal Soc.* A, 113 (1926, 7), 554, Theorem 7.

† I have not been able to prove, by the methods of this paper, the analogous results for a strip without an additional assumption that $\int_{-\infty}^{T} |f(x+iy)|^p \, dy$ is uniformly bounded in $\alpha \leqslant x \leqslant \beta$ for some fixed T. This does not involve any loss of generality under condition (1) as is shown in the paper quoted in the footnote above. But I have not been able to show that this is the case under the less restrictive condition (C) (§ 1.3).

‡ The following line of argument enables us to prove the convexity property. Let $H(\theta) = \int_0^\infty |f(te^{i\theta})|^p \, dt$. Consider the function $H_\epsilon(z)$ of § 3.2, Lemma 2. When ϵ varies in $(0, \infty)$, $\{H_\epsilon(z)\}$ represents, under the conditions of Theorem 3, a uniformly-bounded normal family of sub-harmonic functions. The function $v(z)$ contemplated in § 3.1 (iv) is precisely $H(\theta)$ as defined above. Hence $H(\theta)$

1.2. *Definition and notation.* In what follows, we mean, by a line, a half-line issuing from the origin. We mean, by an angle, one or other of the two infinite regions enclosed by two half-lines issuing from the origin; when we wish to speak of the magnitude of the angle we consider that angle which maps out the region under consideration. Let $f(z)$ be a function regular in the interior and on the sides of an angle Λ. We shall denote by $M(r, \Lambda, f)$ the maximum of $|f(z)|$ in the sector cut out by Λ on the circular region $|z| \leqslant r$.

1.3. We require the following Lemma:*

LEMMA 1. *Let $f(z)$ be a function regular in an angle Λ of magnitude α. Let*†

$$\varlimsup_{r\to\infty} \frac{\log M(r, \Lambda, f)}{r^{\pi/\alpha}} \leqslant 0. \tag{C}$$

Let $|f(z)| \leqslant A$ on the sides of the angle. Then $|f(z)| \leqslant A$ throughout the angle.

2. THEOREM 1. *Let $f(z)$ be a function regular in an angle Λ of magnitude α. Let it satisfy the condition* (C) *in Λ. Let $f(z)$ belong to the class L_p (p a positive integer) along the sides of Λ. Then it belongs to L_p uniformly along every line in Λ.*

Proof. Let $z_0 = r_0 e^{i\theta_0}$ be any fixed point in Λ. Define $P(r)$ for $0 \leqslant r \leqslant r_0$ by the relation

$$P(r) = [\operatorname{sgn} f(re^{i\theta_0})]^{-p}$$

where, as usual,

$$|f(z)|\,[\operatorname{sgn} f(z)] = f(z) \quad (f(z) \neq 0)$$

and

$$\operatorname{sgn} f(z) = 0 \quad (f(z) = 0).$$

Let $z = re^{i\theta}$ and

$$H(z) = z \int_0^1 [f(z\lambda)]^p P(r_0 \lambda)\, d\lambda = e^{i\theta} \int_0^r [f(te^{i\theta})]^p P\left(\frac{r_0}{r} t\right) dt.$$

Now $H(z)$ is a function regular in Λ. By hypothesis it is bounded

is sub-harmonic and constant on the curves $\theta =$ constant. Therefore by (v) § 3.1, $H(\theta)$ is a convex function of θ. By applying this result to $z^{-i\alpha/p} f(z)$ ($\alpha > 0$), we conclude the possibility that $e^{\alpha\theta} H(\theta)$ be convex in θ when $\alpha > 0$. Hence by another result proved by Montel in the paper quoted later we conclude that $\log H(\theta)$ is a convex function of θ.

* R. and F. Nevanlinna, *Ac. Soc. Fennicae*, 50.

† I am indebted to a referee for replacing my original slightly less general and cumbersome conditions by the elegant hypothesis (C) in the Lemma and the succeeding theorems.

along the sides of Λ. Moreover, if $f(z)$ satisfies condition (C), so does $H(z)$. Hence, by Lemma (1), $H(z)$ is uniformly bounded in Λ. But, at $z = z_0$,

$$|H(z_0)| = \int_0^{r_0} |f(te^{i\theta_0})|\, dt.$$

Since z_0 is any point in Λ, the theorem follows.

2.1. THEOREM 2. *Let Γ be a system consisting of a finite number of lines. Let α be the magnitude of the greatest of the angles between consecutive members of the system. Let $f(z)$ be an integral function which satisfies* (C) *in each angle Λ between consecutive members of Γ. Let $f(z)$ belong to some class L_p $(p \geqslant 1)$ on the lines of Γ, where p might be different on the different lines. Then $f(z) \equiv 0$.*

Proof. Let ϵ be any given positive number. Consider the function

$$G_\epsilon(z) = z \int_0^\epsilon f(z\lambda)\, d\lambda = e^{i\theta} \int_0^{r\epsilon} f(te^{i\theta})\, dt,$$

where $z = re^{i\theta}$. $G_\epsilon(z)$ is an integral function and satisfies condition (C), when $f(z)$ does so, in each angle Λ between consecutive members of Γ. Using Schwarz's inequality, we easily find that, whatever $p \geqslant 1$ be on the sides of Λ, $|G_\epsilon(z)/z|$ is bounded along these sides as $|z| \to \infty$. Hence by an easy modification of Lemma 1, we conclude that $|G_\epsilon(z)/z|$ is bounded in each angle Λ between consecutive members of Γ. Therefore $|G_\epsilon(z)/z|$ is bounded in the whole plane as $|z| \to \infty$. Therefore $G_\epsilon(z)$ is a polynomial of degree one at most. Differentiating with respect to ϵ and putting $\epsilon = 1$, we find that $zf(z)$ is of the form $az+h$. Since $f(z)$ is an integral function it must reduce to a constant which is zero since $f(z)$ belongs to some L_p along lines of Γ.

2.2. The following corollaries are easily derived from Theorem 2.

COROLLARY 1. *If the function $f(z)$ be of order ρ less than π/α or if it be of order π/α and minimal type, it will be identically zero under the hypothesis of Theorem* 2.

COROLLARY 2. *If an integral function of order one and minimal type belongs to L_p $(p \geqslant 1)$, along a* WHOLE *line, then it is identically zero.**

For, we can take the origin at some point on the line so that we get two half-lines inclined at an angle π. So we can apply Corollary 1.

* When $p = 2$, the result of Corollary 2 has been proved by Paley and Wiener, *American Math. Soc. Colloq. Pub.* xix, 13, Theorem XI.

COROLLARY 3. *If an integral function of order one belongs to some L_p $(p \geqslant 1)$ on two* INTERSECTING WHOLE LINES *in the plane, then it is identically zero.*

Here, we can take the point of intersection of the two lines as the origin. The greatest angle between the consecutive lines is less than π. Therefore we can apply Corollary 1.

3. We shall use the theory of sub-harmonic functions to extend the above results to the case $p > 0$. We require the following from the theory of sub-harmonic functions:*

3.1. A real continuous function $u(z)$ of z defined in a domain D is said to be 'sub-harmonic in D' when the value of the function at any point z_0 in the interior of D does not exceed the mean value of the function along all circles of sufficiently small radius round z_0;

(i) a sub-harmonic function does not attain its upper bound in the interior of D unless it reduces to a constant;

(ii) if u and v are sub-harmonic in D, so are $u+v$ and cu, where c is a positive constant;

(iii) if $\{u_n(z)\}$ is a uniformly convergent sequence of sub-harmonic functions in D, then the limit function $u(z)$ is sub-harmonic in D;

(iv) let $\{u\}$ be a uniformly-bounded normal family of sub-harmonic functions, and $v(z)$ be the function equal to the greatest limit of the set $\{u(z)\}$ at z where u varies over the functions of the family, then $v(z)$ is sub-harmonic in D;

(v) let $u(z)$ be a harmonic function, and $v(z)$ be a sub-harmonic function taking constant values on the curves $u(z) = \text{constant}$, then v, considered as a function of u, is a convex function of u;

(vi) let $f(z)$ be regular in D, then $|f(z)|^p$ $(p > 0)$ is sub-harmonic in D.

3.2. We next prove the following lemmas:

LEMMA 2. *Let ϵ be a fixed positive number. Let $f(z)$ be a function regular in an angle Λ. Let $z = re^{i\theta}$ and*

$$H_\epsilon(z) = r\int_0^\epsilon |f(z\lambda)|^p\,d\lambda = \int_0^{r\epsilon} |f(te^{i\theta})|^p\,dt \quad (p > 0).$$

Then $H_\epsilon(z)$ is sub-harmonic in Λ. Let $g(z)$ be regular in Λ. Then $|e^{g(z)}H_\epsilon(z)|$ is sub-harmonic in Λ.

* For these properties, see P. Montel, *J. de Math.* (New Series), 7 (1928).

Proof. We have

$$|e^{g(z)}H_\epsilon(z)| = \lim_{n\to\infty}\frac{\epsilon}{n}\sum_{k=1}^{n}\left|z^{1/p}e^{g(z)/p}f\left(\frac{k\epsilon}{n}z\right)\right|^p,$$

uniformly in any finite part of Λ. Hence, by (iii) and (vi) of § 3.1, $|e^{g(z)}H_\epsilon(z)|$ is sub-harmonic. The first part of the lemma follows on taking $g(z) \equiv 0$.

3.3. LEMMA 3. *Let $f(z)$ be regular in an angle Λ of magnitude α and satisfy condition* (C). *Let $H_\epsilon(z)$ be defined as in Lemma* 2. *If $H_\epsilon(z) \leqslant A$ on the sides of Λ, then $H_\epsilon(z) \leqslant A$ in the whole of Λ.*

Proof. By Lemma 2, $|e^{cz}H_\epsilon(z)|$ is sub-harmonic; therefore* $U = \log H_\epsilon(z)$ is sub-harmonic. Let B be any contour lying in Λ. Let $V(z)$ be the function harmonic in the interior of B and equal to $U(z)$ on B. Let G be the Green's function belonging to B. Then

$$V(z) = \frac{1}{2\pi}\int_B V(s)\frac{\partial G}{\partial n}\,ds$$

$$= \frac{1}{2\pi}\int_B U(s)\frac{\partial G}{\partial n}\,ds,$$

where ds is an element of the arc of B. Now $U(z)-V(z)$ is sub-harmonic in the interior of B by (ii) of § 3.1, since $-V(z)$ is sub-harmonic. Also $U(z)-V(z) = 0$ on B. Hence $U(z)-V(z) \leqslant 0$ by (i) of § 3.1 in the interior of B. Hence for z in the interior of B

$$U(z) \leqslant \frac{1}{2\pi}\int_B U(s)\frac{\partial G}{\partial n}\,ds.$$

From this point, the proof now follows that of Lemma 1.

THEOREM 3. *Theorem* 1 *is true for any positive p.*

Proof. We consider the auxiliary functions defined in Lemma 2. By hypothesis the conditions of Lemma 3 are satisfied by $H_\epsilon(z)$, and $H_\epsilon(z) \leqslant A$, some constant, on the sides of Λ. Hence putting $\epsilon = 1$, the theorem follows from Lemma 3.

4.1. THEOREM 4. *Theorem* 2 *remains true when $p > 0$, provided it is the same p for all the lines of Γ.*

Proof. As before, we consider the auxiliary function defined in Lemma 2. By Lemma 3 applied to each of the regions between

* See P. Montel, loc. cit. 40 (footnote 1).

consecutive lines of Γ, we conclude that $H_\epsilon(z)$ is bounded in the whole plane. We, now, require the following

4.2. LEMMA 4. *A sub-harmonic function bounded in the whole plane reduces to a constant.*

Proof. Let $u(z)$ be the sub-harmonic function. Consider the family $\{u(ze^{i\theta})\}$ of sub-harmonic functions, where $0 \leqslant \theta \leqslant 2\pi$. In any circle $|z| \leqslant r$, the family is bounded and normal. Hence, by (iv) of § 3.1, the function equal to the greatest limit of the set $\{u(ze^{i\theta})\}$ $(0 \leqslant \theta \leqslant 2\pi)$ at the point z is sub-harmonic. But this is evidently $M(r)$, where $z = re^{i\theta}$ and $M(r)$ denotes the maximum of $u(z)$ on $|z| = r$. By (i) of § 3.1, $M(r)$ is a non-decreasing function of r. By (v) of § 3.1, $M(r)$ is a convex function of $\log r$. By using the convexity-property of $M(r)$ it is easy to show that $M(r)$ is a constant since it is bounded. Hence $u(0) = M(r)$. We can now apply (i) of § 3.1 and conclude that $u(z)$ is a constant.

4.3. Resuming the proof of Theorem 4, we find that $H_\epsilon(z)$ is a constant by Lemma 4. Differentiating with respect to ϵ and letting $\epsilon = 1$, we get $|z|\,|f(z)|^p$ is a constant which can only be zero since $f(z)$ is an integral function. This completes the proof of Theorem 4.

4.4. *Conclusion.* It might be asked whether an integral function can belong to some L_p along every line. The answer is in the affirmative. We know* that there is a function $g(z)$ tending to zero along every line. The function

$$\frac{g(z)+g(-z)-2g(0)}{z^2}$$

belongs to L_p $(p > \frac{1}{2})$ along every line. But all the same, no function can belong to some L_p *uniformly* along every line unless it is identically zero. For, the auxiliary function $H_\epsilon(z)$ of Lemma 2 would be bounded in the whole plane so that by Lemma 3 it reduces to a constant. Hence, as before, $f(z) \equiv 0$.

* See P. Dienes, *Taylor Series* (Oxford, 1931), Chap. X, 346.

NOTE ON A THEOREM OF HELMHOLTZ

By J. HODGKINSON (*Oxford*)

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A FLUID occupying infinite space is at rest at infinity. Within a closed surface S there are an expansion θ and vorticities (ξ, η, ζ); θ, ξ, η, ζ are everywhere zero outside S. Then the velocity components (u, v, w) at a point (x, y, z) are given by Helmholtz's formulae

$$u = -\frac{\partial\Phi}{\partial x}+\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z},$$

$$v = -\frac{\partial\Phi}{\partial y}+\frac{\partial F}{\partial z}-\frac{\partial H}{\partial x},$$

$$w = -\frac{\partial\Phi}{\partial z}+\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}.$$

Here

$$\Phi = \frac{1}{4\pi}\iiint \frac{\theta'}{r}\, dx'dy'dz',$$

$$F = \frac{1}{2\pi}\iiint \frac{\xi'}{r}\, dx'dy'dz',$$

$$G = \frac{1}{2\pi}\iiint \frac{\eta'}{r}\, dx'dy'dz',$$

$$H = \frac{1}{2\pi}\iiint \frac{\zeta'}{r}\, dx'dy'dz';$$

the integrations are taken throughout the space enclosed by S, and r is the distance between the point (x,y,z) and the point (x',y',z') at which the volume-element of the integral is situate.*

It has been suggested that, although there is no doubt that any fluid motion may be represented by the formulae, some restrictions must be placed on the values of θ, ξ, η, ζ in order that the formulae may represent a fluid motion. This note provides a discussion of this point.

The formulae may be interpreted in two ways. If they are required to represent a fluid motion for all time, i.e. θ, ξ, η, ζ are functions of the time t as well as of the coordinates, it seems that the formulae are nothing but an alternative presentation of the fundamental

* H. von Helmholtz, *J. für Math.* 55 (1858), 25–55. The notation used is that of H. Lamb, *Hydrodynamics* (6th ed. 1932), Chap. VII, 208–10.

differential equations of hydrodynamics. But we may regard the formulae as representing a fluid motion for one instant only, say the initial moment $t = 0$. With this interpretation θ, ξ, η, ζ are functions of the coordinates only. For precision of statement we replace the surface S by two surfaces S_1, S_2. Within S_1 there is no volume throughout which θ vanishes; within S_2 there is no volume throughout which ξ, η, ζ all vanish. I shall suppose that θ, and ξ, η, ζ are regular functions within S_1 and S_2, and, of course, that

$$\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}+\frac{\partial \zeta}{\partial z} = 0.$$

I shall also suppose that the fluid pressure p is a function of the density ρ, and that the body forces may be derived from a potential. I shall show that the only further restriction necessary in order that Helmholtz's formulae may represent a fluid motion at an 'ordinary moment' is that the value of θ must tend to zero at all points on S_1, when these points are approached from within S_1. [By the phrase 'ordinary moment' I mean a moment at which the acceleration at a point of the fluid is everywhere continuous.]

A very brief discussion of the first interpretation of the formulae suffices. Within S_1, S_2 the values of θ, ξ, η, ζ are those of a fluid motion. They must satisfy the relations*

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\theta = 0 \tag{A}$$

$$\left.\begin{aligned}
\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) &= \frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial u}{\partial y}+\frac{\zeta}{\rho}\frac{\partial u}{\partial z}\\
\frac{D}{Dt}\left(\frac{\eta}{\rho}\right) &= \frac{\xi}{\rho}\frac{\partial v}{\partial x}+\frac{\eta}{\rho}\frac{\partial v}{\partial y}+\frac{\zeta}{\rho}\frac{\partial v}{\partial z}\\
\frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) &= \frac{\xi}{\rho}\frac{\partial w}{\partial x}+\frac{\eta}{\rho}\frac{\partial w}{\partial y}+\frac{\zeta}{\rho}\frac{\partial w}{\partial z}.
\end{aligned}\right\} \tag{B}$$

But, since

$$(\xi,\eta,\zeta) = \tfrac{1}{2}\,\mathrm{curl}(u,v,w), \qquad \theta = \mathrm{div}(u,v,w),$$

these may be regarded as three differential relations satisfied by ρ, u, v, w in any fluid motion. [The equations (B) are not independent. If they are differentiated with respect to x, y, z respectively, the equation found by adding the differentiated relations is satisfied

* E. J. Nansen, *Messenger of Math.* 3 (1874), 120; Lamb, loc. cit. 205.

identically in virtue of (A) and the relation

$$\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}+\frac{\partial \zeta}{\partial z} = 0.$$

They have the same kind of dependence as the equations $\partial z/\partial x = P$, $\partial z/\partial y = Q$, where $\partial Q/\partial x = \partial P/\partial y$.] It seems therefore that in order to find θ, ξ, η, ζ so that Helmholtz's formulae may represent a fluid motion, we must determine a fluid motion to which θ, ξ, η, ζ belong. In other words, we have merely restated the analytical problem presented by the differential equations of fluid motion.

We turn to the second interpretation of the formulae. The integrals Φ, F, G, H are of the same form as the integral defining the gravitational potential of a volume-distribution of mass, and the first derivatives of such a potential function are continuous everywhere. Hence u, v, w are continuous everywhere. Thus the motion defined by the formulae is kinematically possible. This is the sense in which I interpret some sentences of Lamb's account.*

We cannot, however, ignore dynamical considerations altogether. The differential equations of fluid motion are formed with the assumption that the pressure-gradient is continuous,† and consequently so are the accelerations of the particles of fluid. It is conceivable that, at a 'singular moment' of the motion, the acceleration might become discontinuous at the points of some surface within the fluid without the collapse of the motion. It might well be true that Helmholtz's formulae would represent a motion at a singular moment unless the values of θ, ξ, η, ζ were restricted in some way.

It is clear that, if we show the accelerations of the particles of fluid to be continuous at points on S_1, S_2, we need consider them at no other points. To do this we shall require to make use of the theorem concerning the discontinuity of the normal component of the attraction of a surface-distribution of matter, and the corresponding theorem concerning the discontinuity of the second derivatives of the potential of a volume-distribution.‡ [I state the latter, which is less familiar.] Suppose the density of the distribution within a surface S is ρ. Let V_e be the potential at points outside S, and V_i

* Loc. cit. 204, 206.

† In forming the equations (B) we assume the pressure-gradient differentiable.

‡ G. Kirchhoff, *Vorlesungen über mathematische Physik: Mechanik* (dritte Auflage 1883), 178.

the potential at points inside S, and let l, m, n be the direction-cosines of the outward-drawn normal at a point P on S. Then, at P,

$$\frac{\partial^2 V_e}{\partial x^2} - \frac{\partial^2 V_i}{\partial x^2} = 4\pi l^2 \rho,$$

$$\frac{\partial^2 V_e}{\partial x \partial y} - \frac{\partial^2 V_i}{\partial x \partial y} = 4\pi l m \rho.$$

It will be found necessary to assume the existence of initial values of $\partial\theta/\partial t$, $\partial\xi/\partial t$, $\partial\eta/\partial t$, $\partial\zeta/\partial t$ at points within S_1, S_2. The justification of these assumptions is postponed.

We consider the x-component of acceleration, i.e.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},$$

where
$$u = -\frac{\partial \Phi}{\partial x} + \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}.$$

For the moment we suppose that θ, ξ, η, ζ are functions of t as well as of the coordinates. Then

$$\frac{\partial^2 \Phi}{\partial x \partial t} = \frac{1}{4\pi}\iiint \frac{\partial \theta'}{\partial t}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) dx'dy'dz' + \frac{1}{4\pi}\iint v_\nu \theta' \frac{\partial}{\partial x}\left(\frac{1}{r}\right) dS_1.$$

The second term of the right-hand member represents that part of the rate of variation of $\partial\Phi/\partial x$ which is due to the motion of the surface S_1. This surface will not move with the fluid, in general, and v_ν is written as the velocity of the surface normal to itself. Assuming the existence of an initial value for $\partial\theta/\partial t$, we may now put $t = 0$.

The first term in this expression for $\partial^2\Phi/\partial x \partial t$ can be represented as the x-component of attraction of a volume-distribution of matter. This component is continuous at points on S_1. The second term is similarly represented as the x-component of the attraction of a surface-distribution. The discontinuity of the normal component is $-v_\nu \theta$. Hence the discontinuity of the x-component is $-l v_\nu \theta$, i.e. we may write

$$\frac{\partial u_e}{\partial t} - \frac{\partial u_i}{\partial t} = l v_\nu \theta.$$

We next turn to $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$. The contributions of θ to these quantities are

$$-\frac{\partial^2 \Phi}{\partial x^2}, \qquad -\frac{\partial^2 \Phi}{\partial x \partial y}, \qquad -\frac{\partial^2 \Phi}{\partial x \partial z},$$

which we represent as second derivatives of the potential of a

volume-distribution. By means of the theorem I have quoted we find that

$$\frac{\partial u_e}{\partial x}-\frac{\partial u_i}{\partial x}=-l^2\theta,\qquad \frac{\partial u_e}{\partial y}-\frac{\partial u_i}{\partial y}=-lm\theta,\qquad \frac{\partial u_e}{\partial z}-\frac{\partial u_i}{\partial z}=-ln\theta.$$

Hence the total discontinuity in the x-components of the accelerations of the particles of fluid at points on the surface S_1 is

$$l(v_\nu-lu-mv-nw)\theta.$$

It was noted that, in general, the surface S_1 does not move with the fluid, i.e. $v_\nu-lu-mv-nw$ does not vanish. In order, then, that the accelerations may be continuous at points on S_1, it is necessary that the value of θ shall tend to zero as we approach such points from within S_1.

We consider the possible discontinuities of the accelerations at points on S_2 in the same manner. These are calculated from the integrals F, G, H. The important difference in this calculation is due to the fact that S_2 moves with the fluid, and the factor corresponding to $v_\nu-lu-mv-nw$ above vanishes. Hence the presence of the vortices causes no discontinuity at points on the surface S_2. In this calculation we have assumed the existence of initial values of $\partial\xi/\partial t$, $\partial\eta/\partial t$, $\partial\zeta/\partial t$.

We have now to justify the assumption of initial values of $\partial\xi/\partial t$, $\partial\eta/\partial t$, $\partial\zeta/\partial t$, $\partial\theta/\partial t$.

The equation
$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial u}{\partial y}+\frac{\zeta}{\rho}\frac{\partial u}{\partial z}$$
may be written in the form

$$\frac{\partial\xi}{\partial t}=\xi\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z}-u\frac{\partial\xi}{\partial x}-v\frac{\partial\xi}{\partial y}-w\frac{\partial\xi}{\partial z}-\theta\xi.$$

Helmholtz's formulae give the values of u, v, w at points within S_2, and we are thus able to assign an initial value to $\partial\xi/\partial t$ without further information.

Further
$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z},$$
so that
$$\frac{\partial\theta}{\partial t}=\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 v}{\partial y\partial t}+\frac{\partial^2 w}{\partial z\partial t};$$
also
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=X-\frac{1}{\rho}\frac{\partial p}{\partial x},$$
whence
$$\frac{\partial^2 u}{\partial x\partial t}=\frac{\partial}{\partial x}\left(X-\frac{1}{\rho}\frac{\partial p}{\partial x}-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}-w\frac{\partial u}{\partial z}\right).$$

To justify the assumption of an initial value of $\partial\theta/\partial t$ at points within S_1, we must assume that the pressure p is a twice-differentiable function of ρ, and that the potential of the impressed forces and the initial density of the fluid are twice-differentiable functions of the coordinates. The same conditions are in fact required to establish the dynamical equation

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial u}{\partial y}+\frac{\zeta}{\rho}\frac{\partial u}{\partial z}.$$

These latter conditions, however, do not apply to the prescribed values of θ, ξ, η, ζ. They operate as limitations upon the system whose motion is described by Helmholtz's formulae.

It is now established that, provided that θ tends to zero at points on S_1, the approach to these points being made from within S_1, Helmholtz's formulae represent a motion of a fluid at an ordinary moment in the sense that the accelerations are everywhere continuous. If θ does not satisfy this provision, the formulae represent a fluid motion at a moment which is 'singular' in the sense that the accelerations at points on S_1 are momentarily discontinuous.

HYPERGEOMETRIC PARTIAL DIFFERENTIAL EQUATIONS (II)

By T. W. CHAUNDY (*Oxford*)

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1. Euler's equation

In a previous paper under this title (**3**) I discussed the solution of the partial differential equation

$$f(\delta)F(\delta')V = xy\,g(\delta)G(\delta')V \qquad \left(\delta, \delta' \equiv x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right)$$

and its extension to fields of more independent variables, obtaining their 'general solutions' in terms of 'fundamental solutions' expressed as multiple power-series of hypergeometric type.* In this new paper, presupposing the same structure for the general solution (which is essentially Riemann's), I obtain an alternative expression for the fundamental solution which is simpler in practice and has some advantages from the theoretical point of view.

I begin with Euler's equation

$$(x-y)\frac{\partial^2 V}{\partial x \partial y}+n\frac{\partial V}{\partial x}-m\frac{\partial V}{\partial y} = 0 \tag{1}$$

of which the solution is known:† or rather with the variant

$$\delta\delta' V = xy(\delta-m)(\delta'-n)V \tag{2}$$

obtained from (1) by writing $x^m V$ for V and x^{-1} for x (or, of course, by the similar transformation in y). The adjoint of (2), in the operators δ, δ', is

$$\delta\delta' U = xy(\delta+m+1)(\delta'+n+1)U. \tag{3}$$

The fundamental solution $U(X, Y; x, y)$ is a solution of (3) such that‡

(i) on $x = X$, it satisfies $\delta' U = Xy(\delta'+n+1)U$, i.e. it is of the form $U = C_1(1-Xy)^{-(n+1)}$;

(ii) on $y = Y$, it satisfies $\delta U = xY(\delta+m+1)U$, i.e. it is of the form $U = C_2(1-xY)^{-(m+1)}$;

(iii) at $x = X$, $y = Y$, it reduces to $(1-XY)^{-1}$.

* **3,** 290 (8); or more precisely 293, § 3.

† **4,** chap. 3, 54–70; chap. 4, 81–4.

‡ **3,** 294 (21), (23).

The last condition fixes the constants C_1, C_2, and we can state the conditions more compactly thus:

$$U = (1-XY)^n/(1-Xy)^{n+1} \quad \text{on } x = X, \tag{4}$$

$$U = (1-XY)^m/(1-xY)^{m+1} \quad \text{on } y = Y. \tag{5}$$

I now suppose m, n positive integers and write

$$U \equiv (\delta'+1)(\delta'+2)...(\delta'+n)W,$$

so that (3) becomes

$$\delta'(\delta'+1)...(\delta'+n)\{\delta W-xy(\delta+m+1)W\} = 0.$$

A possible solution is given by

$$\delta W-xy(\delta+m+1)W = 0,$$

i.e. by
$$W = \phi(y)(1-xy)^{-(m+1)},$$

where ϕ is an arbitrary function. Thus a possible solution of (3) is

$$\begin{aligned} U &= (\delta'+1)(\delta'+2)...(\delta'+n)\{\phi(y)(1-xy)^{-(m+1)}\} \\ &= y^{-n}\delta'(\delta'-1)...(\delta'-n+1)\{y^n\phi(y)(1-xy)^{-(m+1)}\} \\ &= \frac{\partial^n}{\partial y^n}\{y^n\phi(y)(1-xy)^{-(m+1)}\}. \end{aligned}$$

Changing the arbitrary function ϕ, write this

$$U = \frac{\partial^n}{\partial y^n}\left\{\frac{(y-Y)^n}{n!}\psi(y)(1-xy)^{-(m+1)}\right\}.$$

Then, on $y = Y$, $\quad U = \psi(Y)(1-xY)^{-(m+1)}.$

Thus (5) is satisfied if we take

$$\psi(y) \equiv (1-Xy)^m.$$

Let us try, then,

$$U(X,Y;\, x,y) = \frac{\partial^n}{\partial y^n}\left\{\frac{(y-Y)^n}{n!}(1-Xy)^m(1-xy)^{-(m+1)}\right\}. \tag{6}$$

On $x = X$ it reduces to

$$\begin{aligned} U &= \frac{\partial^n}{\partial y^n}\left\{\frac{(y-Y)^n}{n!}(1-Xy)^{-1}\right\} \\ &= \frac{\partial^n}{\partial y^n}\left\{P_{n-1}(y)+\frac{(X^{-1}-Y)^n}{n!(1-Xy)}\right\}, \end{aligned}$$

where $P_{n-1}(y)$ is a polynomial of degree $n-1$. Hence, on $x = X$,

$$U = (1-XY)^n/(1-Xy)^{n+1},$$

which satisfies (4). Thus (6) is the required fundamental solution.

Working in x instead of y, we should similarly have obtained

$$U(X,Y;x,y)=\frac{\partial^m}{\partial x^m}\left\{\frac{(x-X)^m}{m!}(1-xY)^n(1-xy)^{-(n+1)}\right\}. \tag{7}$$

Finally, we can show by similar arguments that

$$U(X,Y;x,y)=\frac{\partial^{m+n}}{\partial x^m\partial y^n}\left\{\frac{(x-X)^m(y-Y)^n}{m!\,n!(1-xy)}\right\} \tag{8}$$

is a solution of (3) and satisfies the conditions (4), (5). It is not difficult, of course, to verify that (6), (7), (8) are the same function.

We can identify (6) with the known fundamental solution*

$$U=(1-xy)^{-(m+n+1)}(1-Xy)^m(1-xY)^n\times$$
$$\times F\left[-m,-n;1;\frac{(x-X)(y-Y)}{(1-xY)(1-Xy)}\right], \tag{9}$$

where F is the elementary hypergeometric function, if we notice from (6) that $(1-xy)^nU$ is, by Taylor's formula, the coefficient of h^n in the expansion (in ascending powers of h) of

$$\frac{\{1-X[y+h(1-xy)]\}^m\{y+h(1-xy)-Y\}^n}{\{1-x[y+h(1-xy)]\}^{m+1}}$$
$$=\frac{\{h(x-X)+(1-Xy)(1-hx)\}^m\{(y-Y)(1-hx)+h(1-xY)\}^n}{(1-xy)^{m+1}(1-hx)^{m+1}}.$$

Thus $(1-xy)^{m+n+1}U$ is the coefficient of h^n in

$$\sum_{s=0}\sum_{t=0}(m!s)(n!t)(x-X)^s(y-Y)^t(1-Xy)^{m-s}(1-xY)^{n-t}h^{n+s-t}(1-hx)^{t-s-1},$$

where $(m!s)$, $(n!t)$ denote binomial coefficients. Now the terms in which $s<t$ are polynomials in h of degree $n-1$ and so do not contain h^n: nor do the terms in which $s>t$. Thus, at length,

$$U=(1-xy)^{-(m+n+1)}\sum_{t=0}(m!t)(n!t)(x-X)^t(y-Y)^t(1-Xy)^{m-t}(1-xY)^{n-t}$$
$$=(1-xy)^{-(m+n+1)}(1-Xy)^m(1-xY)^n\times$$
$$\times F\left[-m,-n;1;\frac{(x-X)(y-Y)}{(1-Xy)(1-xY)}\right],$$

which is (9).

* **4,** 83 (22) gives the fundamental solution of Euler's equation, which is equivalent, by elementary transformations, to (9).

2. The fundamental solution as a contour integral

These forms for the fundamental solution become more significant when expressed as contour integrals. We may write (6) as

$$2\pi i U(X,Y;\,x,y) = \int\limits_C \frac{(1-Xt)^m(t-Y)^n\,dt}{(1-xt)^{m+1}(t-y)^{n+1}}, \tag{10}$$

where C is a simple contour including $t = y$ but excluding $t = x^{-1}$. If we write $t = \tau^{-1}$, this becomes

$$2\pi i U = \int\limits_{C'} \frac{(\tau-X)^m(1-\tau Y)^n\,d\tau}{(\tau-x)^{m+1}(1-\tau y)^{n+1}}, \tag{11}$$

where C' is a simple contour including $\tau = x$ but excluding $\tau = y^{-1}$.* This is evidently the form we get by converting (7) into a contour integral: the symmetry of (10) in x's and y's is therefore clear.

We now remark that

$$u = (1-xy)^{-(m+1)}, \qquad v = (1-xy)^m$$

are the solutions of the adjoint pair of ordinary differential equations (in x)

$$\delta u = xy(\delta+m+1)u, \qquad \delta v = xy(\delta-m)v; \tag{12}$$

while

$$u = (1-xy)^{-(n+1)}, \qquad v = (1-xy)^n$$

are similarly the solutions of the adjoint pair (in y)

$$\delta' u = xy(\delta'+n+1)u, \qquad \delta' v = xy(\delta'-n)v. \tag{13}$$

If we write these four solutions (in order) as

$$\xi(xy), \quad \bar{\xi}(xy), \quad \eta(xy), \quad \bar{\eta}(xy),$$

the fundamental solution (10) can be written as

$$2\pi i U(X,Y;\,x,y) = \int\limits_C \xi(xt)\bar{\xi}(Xt)\eta(y/t)\bar{\eta}(Y/t)\frac{dt}{t}, \tag{14}$$

where C is a contour including $t = 0, y$ and excluding $t = x^{-1}, \infty$. The arbitrary constants of integration which multiply the solutions $\xi, \eta, \ldots$ in (14) are presumed to have been suitably chosen.

Passing on now to the general equation in the field of two independent variables

$$f(\delta)F(\delta')V = xy\,g(\delta)G(\delta')V \tag{15}$$

and its adjoint equation

$$f(-\delta)F(-\delta')U = xy\,g(-\delta-1)G(-\delta'-1)U, \tag{16}$$

* Provided that C includes the origin and excludes 'infinity'.

we construct, on the analogy of (14) and with the same contour C, the function

$$2\pi i U(X,Y;x,y) = \int\limits_C \Big\{\sum_r \xi_r(xt)\bar{\xi}_r(Xt)\Big\}\Big\{\sum_s \eta_s(y/t)\bar{\eta}_s(Y/t)\Big\}\frac{dt}{t}, \quad (17)$$

where $\{\xi_r(x)\}$, $\{\bar{\xi}_r(x)\}$ are the sets of solutions (suitably paired and with suitable constant multipliers) of the adjoint pairs of ordinary equations in x

$$f(-\delta)u = xg(-\delta-1)u, \qquad f(\delta)v = xg(\delta)v; \quad (18)$$

and $\{\eta_s(y)\}$, $\{\bar{\eta}_s(y)\}$ are similarly chosen sets of solutions of the adjoint pair of equations in y

$$F(-\delta')u = y\,G(-\delta'-1)u, \qquad F(\delta')v = y\,G(\delta')v. \quad (19)$$

Now it is known* that, if $\bar{\xi}_1,\ldots, \bar{\xi}_n$ are a linearly independent set of solutions of a linear ordinary differential equation

$$(\alpha_0\delta^n+\alpha_1\delta^{n-1}+\ldots+\alpha_n)v = 0,$$

then a linearly independent set of solutions of the adjoint equation is given by

$$\alpha_0\xi_1 = \frac{(\delta,1)^{n-2}|\bar{\xi}_2,\ldots,\bar{\xi}_n|}{(\delta,1)^{n-1}|\bar{\xi}_1,\ldots,\bar{\xi}_n|}, \quad \text{etc.},$$

the symbols denoting Wronskians formed with δ. If the solutions ξ_r are chosen in this way, we have the n relations

$$\sum \xi_r\delta^h\bar{\xi}_r = 0 \quad (h = 0,1,\ldots,n-2),$$
$$\sum \xi_r\delta^{n-1}\bar{\xi}_r = \alpha_0^{-1},$$

and the n reciprocal relations

$$\left.\begin{aligned}\sum \bar{\xi}_r\delta^h\xi_r &= 0 \quad (h = 0,1,\ldots,n-2)\\ \sum \bar{\xi}_r\delta^{n-1}\xi_r &= (-)^{n-1}\alpha_0^{-1}\end{aligned}\right\}. \quad (20)$$

If now the ξ_r, $\bar{\xi}_r$, η_s, $\bar{\eta}_s$ of (17) are associated in the same way, the relation (20) enables us to show that the fundamental solution U has the proper values on each characteristic $x = X$ and $y = Y$, and at their intersection $x = X$, $y = Y$.

If p, q, P, Q are the orders of the operators $f(\delta)$, $g(\delta)$, $F(\delta')$, $G(\delta')$, these proper values are as follows:†

on $x = X$,

$$\delta^h U = 0 \quad (h < p-1), \quad (21)$$

$$F(-\delta')\delta^{p-1}U = 0 \quad (p > q), \quad (22)$$

$$\{F(-\delta')-Xy\,G(-\delta'-1)\}\delta^{p-1}U = 0 \quad (p = q), \quad (23)$$

with corresponding values on $y = Y$;

* See, for instance, **4**, 115 (12); or, in effect, **2**, 382 (25). The change from the standard operator D to δ presents no difficulty.

† **3**, 294 (19)–(23).

at $x = X, y = Y$,

$$(-\delta)^{p-1}(-\delta')^{P-1}U = \begin{cases} 1 & (p > q \text{ or } P > Q), & (24) \\ (1-XY)^{-1} & (p = q;\ P = Q). & (25) \end{cases}$$

Now on $x = X$ we have, by differentiation across the sign of integration,

$$2\pi i \delta^h U = \int_C \{\sum \delta^h \xi_r(xt)\bar{\xi}_r(Xt)\}\{\sum \eta_s(y/t)\bar{\eta}_s(Y/t)\}\frac{dt}{t}$$

$$= 0, \text{ by (20), if } h < p-1\text{: this gives (21);}$$

and, again by (20), if $h = p-1$,

$$2\pi i(-\delta)^{p-1}U = \int_C \frac{1}{\alpha_0(Xt)}\{\sum \eta_s(y/t)\bar{\eta}_s(Y/t)\}\frac{dt}{t},$$

where $\alpha_0(x)$ is the leading coefficient in (18). Thus, if $p = q$, so that $\alpha_0(x) = 1-x$,

$$2\pi i[(-\delta)^{p-1}U]_{x=X} = \int_C \{\sum \eta_s(y/t)\bar{\eta}_s(Y/t)\}\frac{dt}{t(1-Xt)}$$

$$= \int_{C'} \sum \{\eta_s(y\tau)\bar{\eta}_s(Y\tau)\}\frac{d\tau}{\tau - X} \quad (\tau \equiv t^{-1}),$$

i.e. $$[(-\delta)^{p-1}U]_{x=X} = \sum \eta_s(Xy)\bar{\eta}_s(XY).$$

Since, by definition, every $\eta_s(Xy)$ is a solution of

$$\{F(-\delta')-Xy\,G(-\delta'-1)\}u = 0, \tag{26}$$

we see that (23) is satisfied.

If $p > q$, the leading term $\alpha_0(x)$ is now 1, which can be obtained by applying the limit $x \to 0$ in the previous value. This limit in (26) gives $[(-\delta)^{p-1}U]_{x=X}$ to be a solution of $F(-\delta')u = 0$, so that (22) is satisfied.

Similarly, at the intersection $x = X$, $y = Y$, if $\beta_0(y)$ is the leading coefficient in (19),

$$2\pi i(-\delta)^{p-1}(-\delta')^{P-1}U = \int_C \frac{dt}{t\,\alpha_0(Xt)\beta_0(Y/t)}$$

$$= \int_C \frac{dt}{t(1-Xt)(1-Y/t)}, \quad \text{if } p = q,\ P = Q,$$

so that $(-\delta)^{p-1}(-\delta')^{P-1}U = (1-XY)^{-1}$: which is (25). If $p > q$ or $P > Q$, we get the appropriate result by taking one of the limits $x \to 0$ or $y \to 0$, which then gives (24).

We have still to show that U itself is a solution of the adjoint (16). Now, as regards the arguments x, y only, U is sufficiently written

$$U = \int\limits_C \xi(xt)\eta(y/t)\frac{dt}{t},$$

where $u = \xi$ and $u = \eta$ are solutions of the u-equations in (18) and (19). Thus, by differentiation across the sign of integration,

$$\begin{aligned} f(-\delta)F(-\delta')U &= \int\limits_C \{f(-\delta)\xi(xt)\}\{F(-\delta')\eta(y/t)\}\frac{dt}{t} \\ &= \int\limits_C (xt)(y/t)\{g(-\delta)\xi(xt)\}\{G(-\delta')\eta(y/t)\}\frac{dt}{t} \\ &= xy\,g(-\delta)G(-\delta')U. \end{aligned}$$

This gives U to be a solution of (16) and completes the proof that (17) is the fundamental solution of the adjoint pair of equations (15), (16).

3. Comparison of forms for the fundamental solution

I say 'the' fundamental solution because a theorem can be proved that two solutions $U(X,Y;x,y)$ of (16) are identical if they simultaneously obey all the 'characteristic' conditions (21)–(25). For this reason we can identify the expression of U as a contour integral given by (17) above with the expression of U as a multiple power-series given in (**3**). Let us distinguish the two forms as U_0 (the series) and U_1 (the integral). The formal identification of U_0 and U_1 can be effected by using results due to Burchnall (**1**). I modify his notation to accord with that of (**3**) and I use δ not D as the fundamental operator. Then, as in (**1**), we get for typical solutions of (18), (19)

$$\xi(x) = x^{-a}\sum_{r=0}^{\infty} a_r x^r, \qquad \bar{\xi}(x) = x^{a}\sum_{s=0}^{\infty} \bar{a}_s x^s,$$

$$\eta(y) = y^{-A}\sum_{R=0}^{\infty} A_R y^R, \qquad \bar{\eta}(y) = y^{A}\sum_{S=0}^{\infty} \bar{A}_S y^S,$$

where a, A are typical zeros of $f(\delta)$, $F(\delta)$, and

$$a_r = \prod_{t=0}^{r-1}\frac{g(a-t-1)}{f(a-t-1)}, \qquad \bar{a}_s = \prod_{t=0}^{s-1}\frac{g(a+t)}{f(a+t+1)}, \text{ etc.} \tag{27}$$

Thus, not forgetting the factors μ in (6), (7) of (1) we write

$$2\pi i U_1 = \int_C \sum_a \sum_A \left\{\frac{\xi(xt)\bar{\xi}(Xt)\eta(y/t)\bar{\eta}(Y/t)}{f'(a)F'(A)}\right\}\frac{dt}{t}.$$

Applying the calculus of residues we expand the expression in { }, formally, in powers of t, retaining only the absolute term. Since a typical index of t is $r+s-R-S$, we write

$$r+s = R+S \equiv n, \text{ say,}$$

and sum 'diagonally', in r, s, and again in R, S. This gives

$$U_1 = \sum_a \sum_A \left(\frac{x}{X}\right)^{-a}\left(\frac{y}{Y}\right)^{-A} \sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{S=0}^{n} \frac{\bar{a}_s a_{n-s}\bar{A}_S A_{n-S}}{f'(a)F'(A)} X^s x^{n-s} Y^S y^{n-S}.$$

By (27) we can rewrite

$$\frac{\bar{a}_s a_{n-s}}{f'(a)} = \frac{\prod_{t=0}^{n-1} g(a+s-1-t)}{\prod_{t=0}^{n}{}' f(a+s-t)}, \text{ etc.,}$$

where $\prod'$ indicates that the factor given by $t = s$ is $f'(a)$ instead of $f(a)$. This completes the identification of U_0 and U_1.

The form U_1 is free from some of the limitations imposed on U_0 in (3). In the first place U_1 is applicable to the 'logarithmic' case in which zeros of f or F are equal or differ by an integer. In U_0 the series must be reconstructed with logarithmic terms, but U_1 retains its form provided, of course, that the proper expressions (now involving logarithms) are put for the solutions of the associated ordinary equations (18), (19).

Again, we can adapt the form U_1 to equations of the type

$$f(\delta)F(\delta')V = xg(\delta)G(\delta')V, \tag{28}$$

in which the factor y has disappeared on the right. We could deduce the appropriate form of U_1 by a limiting process, writing y^h for y and then taking the limit $h \to 0$: but we can equally well apply the general arguments of § 2 to (28) as it stands, one pair of the associated ordinary equations being now

$$F(-\delta')u = x\,G(-\delta')u, \qquad F(\delta')v = x\,G(\delta')v, \tag{29}$$

which are satisfied by powers of y with exponents involving x.

As an example consider the adjoint pair

$$\delta\delta' V = x(\delta+a)(\delta+b)V, \qquad \delta\delta' U = x(\delta-a+1)(\delta'-b)U, \tag{30}$$

obtainable as a degenerate form of Euler's equation and its adjoint. We find that the fundamental solution can be written

$$4\pi i U(X,Y;x,y) = \int_{C''} (t-x)^{a-1}(t-X)^{-a}(y/Y)^{-bt/(1-t)}\frac{dt}{1-t}, \tag{31}$$

where C'' is now a 'crossed' contour passing positively round $t = x$ and negatively round $t = 1$. This obeys the 'characteristic' conditions:

on $x = X$, $\quad U \propto y^{-bX/(1-X)}$

and so satisfies the equation

$$\delta' U = X(\delta'-b)U;$$

on $y = Y$, $\quad U \propto (1-x)^{a-1}$

and so satisfies the equation

$$\delta U = x(\delta-a+1)U;$$

at $x = X,\ y = Y$, $\quad U = (1-X)^{-1}$.

4. Extension to more independent variables

The foregoing arguments extend without difficulty to hypergeometric partial differential equations in more than two independent variables. Let us begin with the extension of Euler's equation and its adjoint

$$\left.\begin{aligned}\delta\delta'\delta'' V &= xyz(\delta-m)(\delta'-n)(\delta''-p)V\\ \delta\delta'\delta'' U &= xyz(\delta+m+1)(\delta'+n+1)(\delta''+p+1)U\end{aligned}\right\}, \tag{32}$$

where $\delta'' \equiv z\partial/\partial z$. The appropriate characteristic conditions are these:*

on $x = X$, $\quad \delta'\delta'' U = Xyz(\delta'+n+1)(\delta''+p+1)U$, etc.;

along $y = Y,\ z = Z$, $\quad \delta U = xYZ(\delta+m+1)$, etc.;

at $x = X,\ y = Y,\ z = Z$, $\quad U = (1-XYZ)^{-1}$.

A fundamental solution satisfying these conditions is

$$U(X,Y,Z;x,y,z) = \left(\frac{\partial}{\partial y}\right)^n\left(\frac{\partial}{\partial z}\right)^p \frac{(y-Y)^n(z-Z)^p(1-Xyz)^m}{n!\,p!\,(1-xyz)^{m+1}}. \tag{33}$$

This can be written as a repeated contour integral

$$(2\pi i)^2 U = \int_{C_2} du \int_{C_1} dt \frac{(1-Xtu)^m(t-Y)^n(u-Z)^p}{(1-xtu)^{m+1}(t-y)^{n+1}(u-z)^{p+1}}, \tag{34}$$

* **3**, § 6, 302 (top of the page).

where C_1 is a contour in the t-plane including $t = y$ and excluding $t = (xu)^{-1}$, and C_2 is a contour in the u-plane including $u = z$ and excluding $u = x^{-1}$.

For the general hypergeometric equation in three independent variables

$$f(\delta)F(\delta')\phi(\delta'')V = xyz\,g(\delta)G(\delta')\psi(\delta'')V \tag{35}$$

and its adjoint

$$f(-\delta)F(-\delta')\phi(-\delta'')U = xyz\,g(-\delta-1)G(-\delta'-1)\psi(-\delta''-1)U, \tag{36}$$

we write down the fundamental solution

$$(2\pi i)^2U = \int\limits_{C_2}\frac{du}{u}\int\limits_{C_1}\frac{dt}{t}\{\sum \xi(xtu)\bar{\xi}(Xtu)\}\{\sum \eta(y/t)\bar{\eta}(Y/t)\}\{\sum \zeta(z/u)\bar{\zeta}(Z/u)\}, \tag{37}$$

where $\xi(x)$, $\bar{\xi}(x)$ are associated pairs of solutions of the ordinary differential equations

$$f(-\delta)U = xg(-\delta-1)U, \qquad f(\delta)V = xg(\delta)V, \tag{38}$$

and so forth, and the contours C_1, C_2 are those already defined.

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ON THE CONNECTIVITY OF SPACES OF POSITIVE CURVATURE*

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WE are concerned with a Riemannian space V_N with positive-definite line-element, complete and analytic in the sense of Myers.† It is of positive curvature, that is to say, the Riemannian curvature associated with every 2-element (or, equivalently, the Gaussian curvature of every 2-space formed by geodesics emanating from a 2-element) is greater than some positive number.

If we take any closed curve C in V_N, the process of parallel propagation around C gives a transformation of the vector space at any point P on C. By well-known properties of parallel propagation, this transformation is orthogonal: thus, if we denote the vector spaces before and after by S, S' and the transformation by T, we have

$$S' = TS, \tag{1}$$

where the determinant $\det T$ is ± 1. If we repeat the process, obtaining the vector space S'', we have

$$S'' = TS' = T^2 S, \tag{2}$$

where $\det T^2 = (\det T)^2 = +1$.

We shall call a curve *positively closed* if $\det T = +1$, and *negatively closed* if $\det T = -1$. Obviously every curve taken twice over is positively closed. The positive or negative character of the closure is not altered by deformation of the curve.

If every closed curve in V_N is positively closed, then V_N is orientable, and conversely.

We shall prove the following theorem:

THEOREM I. *Let V_N be a complete analytic Riemannian space of even dimensionality N, positive-definite line-element, and positive curvature. Then*

(i) *every positively-closed curve in V_N can be continuously contracted to a point;*

* Read at the International Congress of Mathematicians, Oslo, 1936.

† S. B. Myers, *Duke Math. Journal*, 1 (1935), 40.

(ii) *every closed curve in V_N taken twice over can be continuously contracted to a point;*

(iii) *if V_N is orientable, it is simply connected.**

Parts (ii) and (iii) are obvious deductions from (i): hence it will be sufficient to establish (i). This we shall do by proving that, if (i) is not true, a contradiction is implied.

Let C be a positively-closed curve which cannot be contracted to a point. (C may be a curve C_0 taken several times over.) Let L be the lower bound of the lengths of all curves into which C can be deformed. We know that V_N is closed;† hence the limit will be attained, and there will exist a positively-closed curve G of length L, such that C can be continuously deformed into G. Obviously, G must be a geodesic, since otherwise its length could be further reduced, below the lower bound L.

Parallel propagation round G gives an orthogonal transformation T of the N-dimensional vector space at a point on G; since G is positively closed, $\det T = +1$. Let $\bar{S}$ be the $(N-1)$-dimensional space of vectors orthogonal to G. Since vectors remain orthogonal to a geodesic under parallel propagation, this orthogonal vector space undergoes an orthogonal transformation

$$\bar{S}' = \bar{T}\bar{S}. \tag{3}$$

Since the tangent vector to the geodesic is invariant for the transformation T, we have

$$\det \bar{T} = \det T = +1. \tag{4}$$

(If G were negatively closed, we should have $\det \bar{T} = \det T = -1$.)

Now an orthogonal transformation in a vector space of M dimensions leaves a line invariant provided that M is odd: if the determinant of the transformation is positive, there is an invariant unit vector ($\mu'^i = \mu^i$): if the determinant is negative there is a reversed vector ($\mu'^i = -\mu^i$).

If, as we shall now suppose, the dimensionality N of V_N is even, then $(N-1)$ is odd. Then $\bar{T}$ is an orthogonal transformation of positive determinant in a vector space of odd dimensionality. *Hence, if G is a positively-closed geodesic in a space of even dimensionality,*

* (iii) for $N = 2$ was proved by Hadamard, but not from the intrinsic point of view: J. Hadamard, *J. de Math.* 3 (1897), 352.

† Cf. Myers, loc. cit. 42.

there exists at any point P on G a unit vector μ^i which is unchanged as a result of parallel propagation round G. (If G were negatively closed, there would be a unit vector reversed as a result of parallel propagation round G.)

Let us now generate a V_2 by drawing geodesics g through all the points of G in the directions indicated by parallel propagation of this 'invariant' unit vector μ^i, and let us consider the singly-infinite family of curves obtained by giving to all the points of G a constant displacement v along g. Let us take in V_2 a system of Gaussian coordinates (u, v), u being equal to the arc on G and running between the same terminal values $(0, L)$ on all the curves $v =$ constant, which are, of course, closed curves on account of the fact that μ^i is unchanged as a result of parallel propagation round G.

Comparing the length of G with the lengths of neighbouring curves $v =$ constant, we know* that the first variation is zero since the displacement is normal to G, and that the second variation is

$$\delta^2 L = \tfrac{1}{2}\delta v^2 (T - I_1 + I_2), \tag{5}$$

where

$$\left.\begin{aligned} T &= \big[\hat{\eta}\cos\theta(\xi, \hat{\eta})\big]_{u=0}^{u=L}, \\ I_1 &= \int_0^L \bar{\xi}\hat{\eta}\cos\theta(\bar{\xi}, \hat{\eta})\,du, \\ I_2 &= \int_0^L \{\bar{\eta}^2\sin^2\theta(\xi, \bar{\eta}) - R_{ijkl}\xi^i\eta^j\xi^k\eta^l\}\,du; \end{aligned}\right\} \tag{6}$$

here

$$\xi^i = \frac{\partial x^i}{\partial u}, \qquad \eta^i = \frac{\partial x^i}{\partial v}; \tag{7}$$

$\bar{\xi}^i$, $\bar{\eta}^i$ are absolute derivatives with respect to u; $\hat{\eta}^i$ an absolute derivative with respect to v; $\bar{\xi}$, $\bar{\eta}$, $\hat{\eta}$ the magnitudes of these vectors; θ the angles between vectors indicated; and R_{ijkl} the curvature tensor. Since v is the arc on g, and u the arc on G, then η^i, ξ^i are unit vectors, and since g are geodesics, $\hat{\eta}^i = 0$. Also, since $\eta^i = \mu^i$ and this vector is propagated parallel to itself along G, we have $\bar{\eta}^i = 0$. Hence

$$\delta^2 L = -\tfrac{1}{2}\delta v^2 \int_0^L K\,ds, \tag{8}$$

* J. L. Synge, *Proc. London Math. Soc.* 25 (1925), 247–64. There is a misprint in equation (6.21); for $\theta(\xi, \eta)$, read $\theta(\xi, \bar{\eta})$.

the integral being taken round G, and K being the Riemannian curvature of V_N for the 2-element of V_2.* Since, by hypothesis, K is positive, $\delta^2 L < 0$, and hence the length of G can be reduced by variation along g. But this contradicts the assumption that L is the lower bound of all curves into which C can be deformed. Hence part (i) of the theorem is proved, and, as remarked above, (ii) and (iii) are immediate consequences.

It is unfortunate that the restrictions to positive-definite line-element and to even dimensionality deprive the results of interest as far as the physical problem of space-time is concerned. For space-time has an indefinite line-element, and a space-like section of space-time, although it has a positive-definite line-element, is of odd dimensionality. It may be of interest to add the following variational result, valid for spaces of odd dimensionality, although it does not lead to a simple topological consequence.

Let G be a (positively or negatively) closed geodesic in a V_N of positive-definite line-element and positive curvature: N may be odd or even. G may indicate a geodesic G_0 taken more than once over. Starting from a point P on G, parallel propagation round G gives an orthogonal transformation of vectors normal to G, a unit vector μ^i transforming into μ'^i. It will in general be impossible to choose μ^i so that $\mu'^i = \mu^i$. But to any choice of μ^i at a point P, there corresponds an angle ϕ, given by

$$\cos\phi = a_{ij}\mu^i\mu'^j, \tag{9}$$

a_{ij} being the fundamental tensor. If in the flat $(N-1)$-dimensional vector space of vectors normal to G at P we construct Σ the $(N-2)$-dimensional unit sphere with centre at P, then ϕ is the length of a geodesic (great-circle) on Σ joining the points where Σ is cut by the vectors μ^i and μ'^i.

The angle ϕ will have a lower bound α for all choices of μ^i normal to G at P. We may call α the 'twist' of the geodesic. (If N is even and G is positively closed, $\alpha = 0$.) Let $(\mu^i)_P$ be the unit vector at P which gives this minimum value to ϕ, and let it become $(\mu'^i)_P$ as a result of parallel propagation round G. Let us now propagate $(\mu'^i)_P$ round G in the same sense as before, returning to P with $(\mu''^i)_P$.

* Since it is proved in the paper referred to above that K is also the Gaussian curvature of V_2, we might have obtained (8) as a variational result intrinsic to V_2.

Now let

$$\left.\begin{aligned} \mu^i &= \text{unit vector field along } G\text{, the result of parallel propagation of } (\mu^i)_P,\\ \mu'^i &= \text{unit vector field along } G\text{, the result of parallel propagation of } (\mu'^i)_P. \end{aligned}\right\} \tag{10}$$

Then along G we have $\quad a_{ij}\mu^i\mu'^j = \cos\alpha. \quad (11)$

Let us now define round G a unit vector ν^i by the equation

$$\nu^i = [\mu^i \sin(s\alpha/L) + \mu'^i \sin(\alpha - s\alpha/L)]/\sin\alpha, \tag{12}$$

where s is the arc of G measured from P and L the total length of G. Then ν^i is a unit vector which equals $(\mu'^i)_P$ for $s = 0$ and again $(\mu'^i)_P$ for $s = L$, because μ^i arrives at P with the value $(\mu'^i)_P$: thus ν^i returns unchanged to P. We have then, taking the absolute derivative,

$$\begin{aligned} \bar{\nu}^i &= \alpha[\mu^i \cos(s\alpha/L) - \mu'^i \cos(\alpha - s\alpha/L)]/(L\sin\alpha),\\ \bar{\nu}^2 &= \alpha^2/L^2. \end{aligned} \tag{13}$$

Let us now draw geodesics g through the points of G in the directions indicated by ν^i and consider the curves into which G may be deformed by taking equal lengths v along g. On account of the way in which ν^i has been defined these are closed curves. Passing from G to a neighbouring curve $v =$ constant, the first variation vanishes, and in the second variation (5) we have, as before, $T = I_1 = 0$. We have also

$$\bar{\eta}^2 = \bar{\nu}^2 = \alpha^2/L^2, \qquad \theta(\xi, \bar{\eta}) = \theta(\xi, \bar{\nu}) = \tfrac{1}{2}\pi, \tag{14}$$

and hence the second variation is

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_0^L (\alpha^2 L^{-2} - K)\, ds. \tag{15}$$

Hence we have this result:

THEOREM II. *Let G be a closed geodesic of length L in a Riemannian space with positive-definite line-element and positive curvature, greater than a constant K_0. If the twist α of G is less than $LK_0^{\frac{1}{2}}$, then G can be deformed into a shorter closed curve.*

Since α is necessarily not greater than π, it follows that G can be deformed into a shorter curve if

$$L > \pi K_0^{-\frac{1}{2}}. \tag{16}$$

This fact is, of course, well known in connexion with conjugate points.*

* Cf. J. L. Synge, *Duke Math. Journal*, 1 (1935), 528, for a treatment by means of the second variation.

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